Intersections, Inverses and the Ubiquity of e

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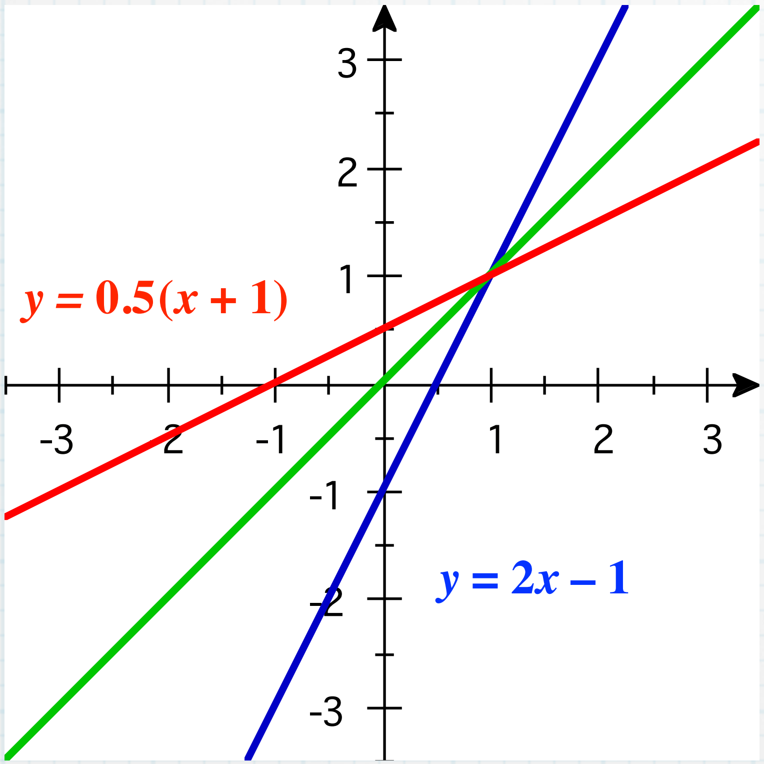
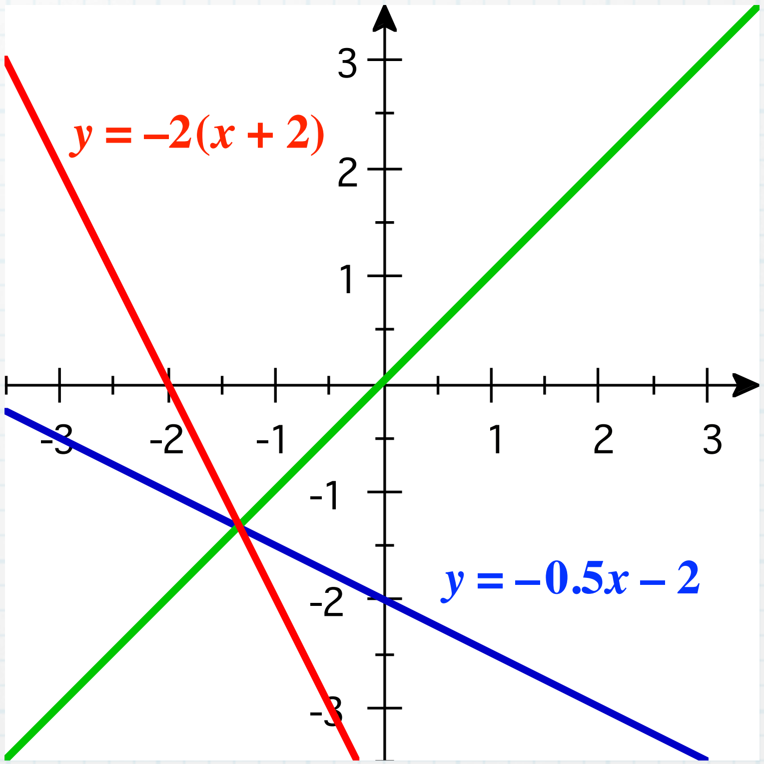
The Graphs of a Function and its Inverse

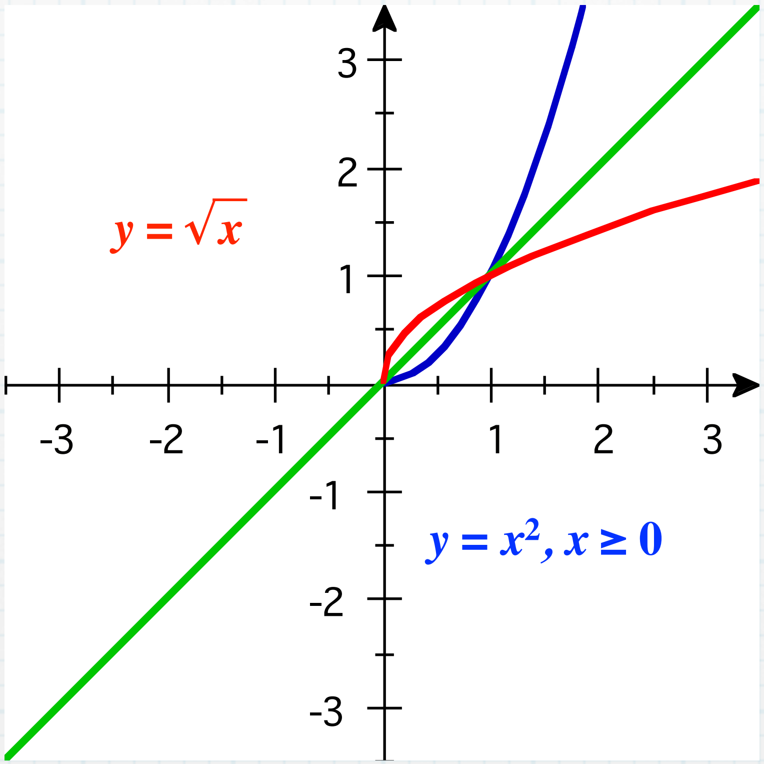
* When a function and its inverse are graphed, are all intersection points on the the line $y = x$?
  * Linear Functions
  * Quadratic Functions
  * Cubic Functions
  * Other, more interesting functions

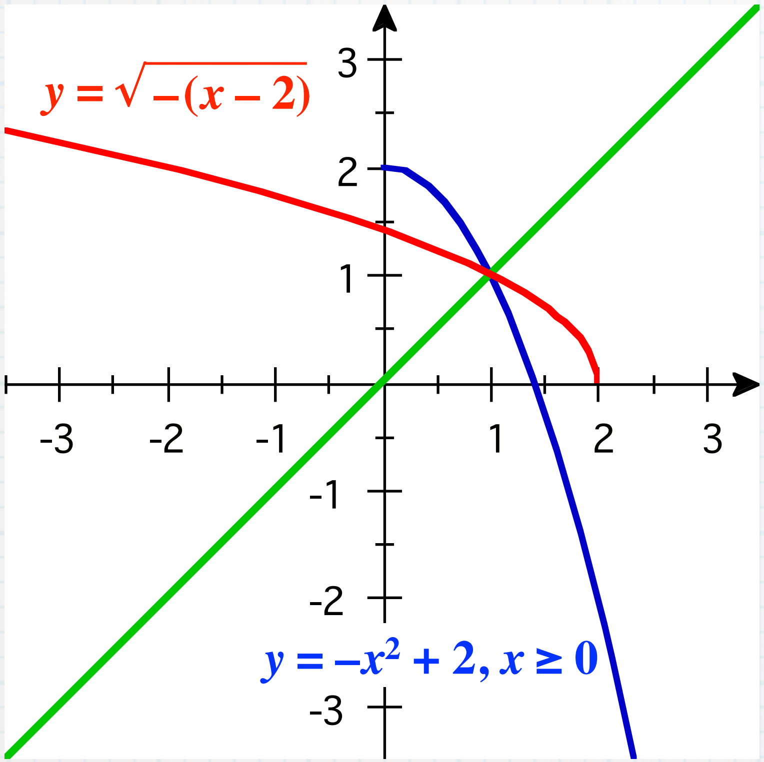
Linear Functions

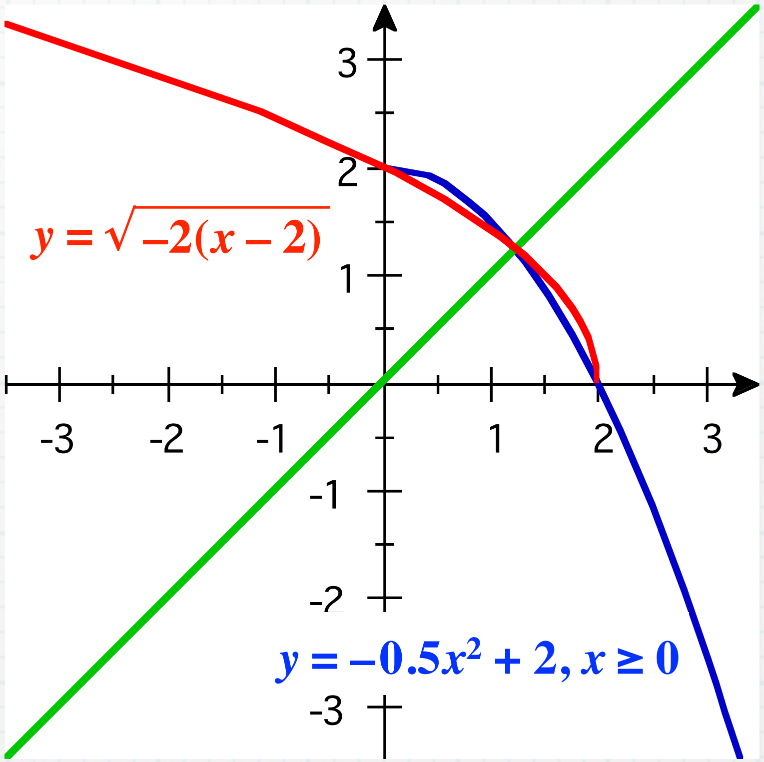




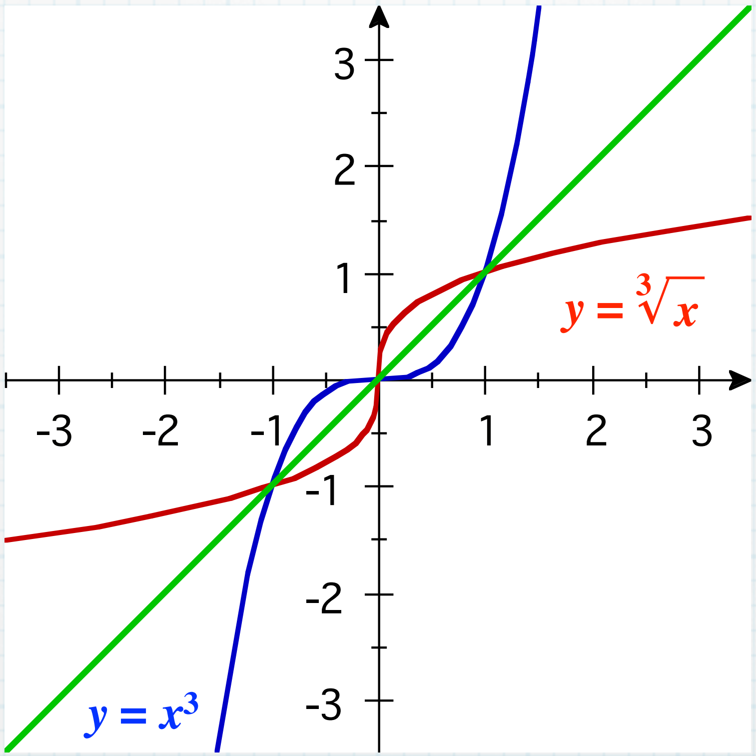
Quadratic Functions




$y = \sqrt{-(x-2)}$
$y = -x^2 + 2, x \geq 0$
3
2
1
-1
-2
-3
-3
-2
-1
1
2
3


$y = \sqrt{-2(x-2)}$
$y = -0.5x^2 + 2, x \geq 0$
3
2
1
-1
-2
-3
-3
-2
-1
1
2
3

Cubic Functions



Let's Build a Cubic Function

We want a cubic function that passes through a point (a, b) not on the line $y = x$ AND passes through the reflection of (a, b) across the line $y = x$, a.k.a. (b, a).

The general form of a cubic function is:

$$y = ax^3 + bx^2 + cx + d$$

Because there are four unknown coefficients, we will need a system of four equations with four unknowns.

Building Our Cubic

$$y = ax^3 + bx^2 + cx + d$$

We will have our cubic function pass through the points $(0, 2)$, $(2, 0)$, $(1, 1)$, and $(-1, -3)$.

Note that the points $(0, 2)$ and $(2, 0)$ are reflections of each other across the line $y = x$.

For $(0, 2)$: $2 = d$

For $(2, 0)$: $0 = 8a + 4b + 2c + d$

For $(1, 1)$: $1 = a + b + c + d$

For $(-1, -3)$: $-3 = -a + b - c + d$

The Solution to this System

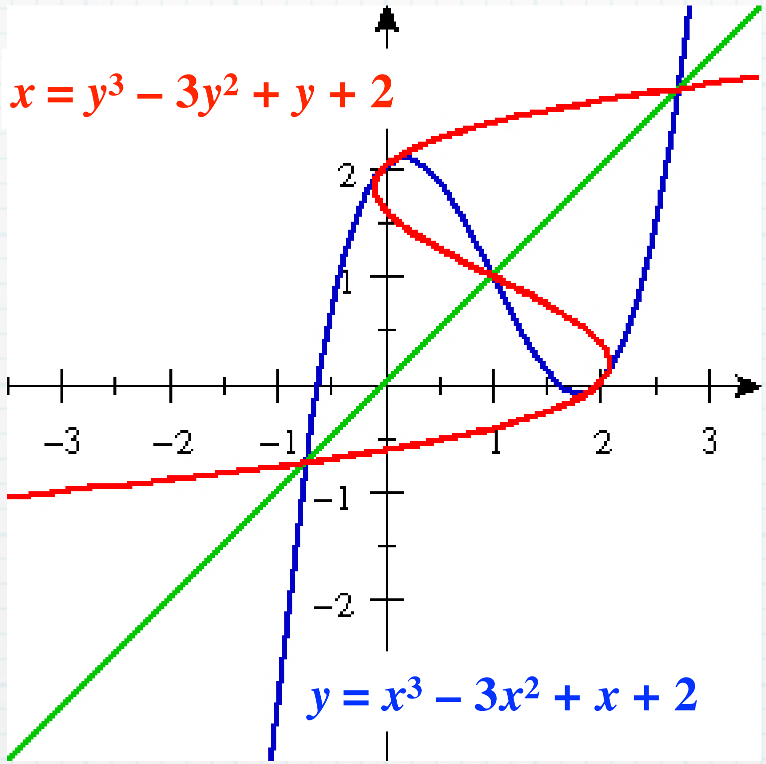
The values of the coefficients are:

$a = 1$, $b = -3$, $c = 1$, and $d = 2$

and our equation is:

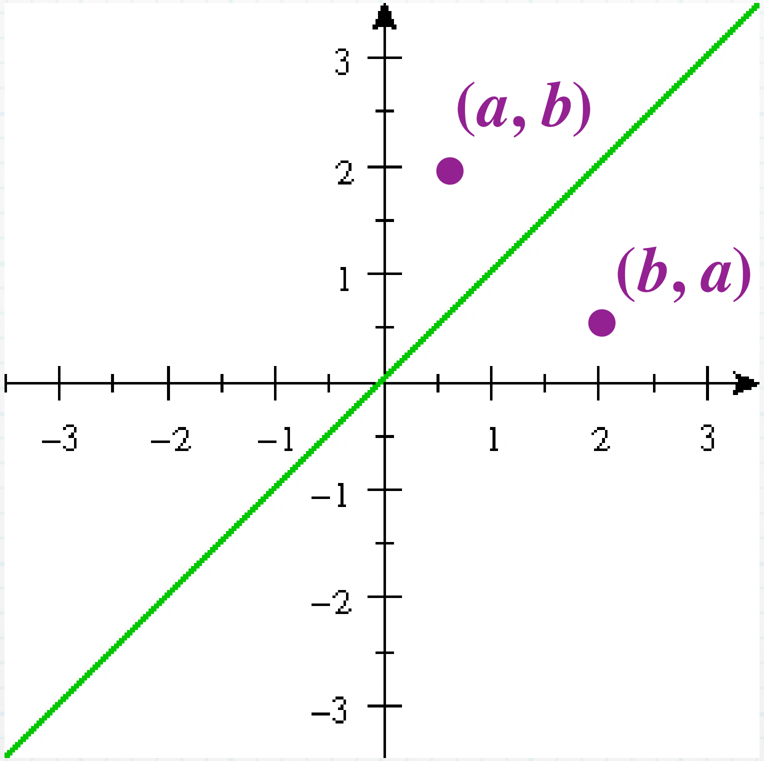
$$y = x^3 - 3x^2 + x + 2$$

So the graph of $y = x^3 - 3x^2 + x + 2$ passes through the points $(2, 0)$ and $(0, 2)$, and so does the graph of its inverse. This should be fun!



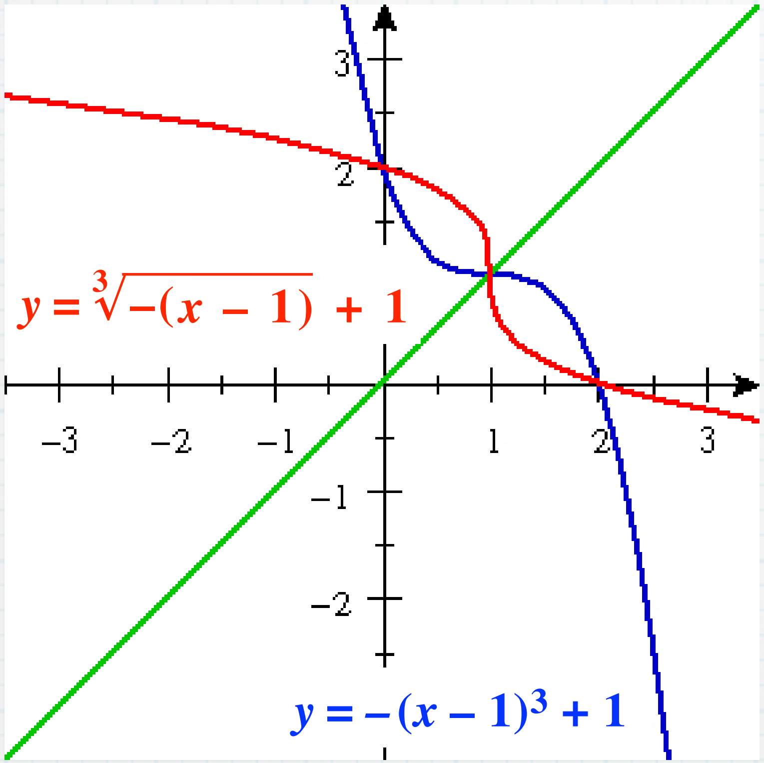
$x = y^3 - 3y^2 + y + 2$
2
1
-1
-2
-3
-2
-1
1
2
3
$y = x^3 - 3x^2 + x + 2$

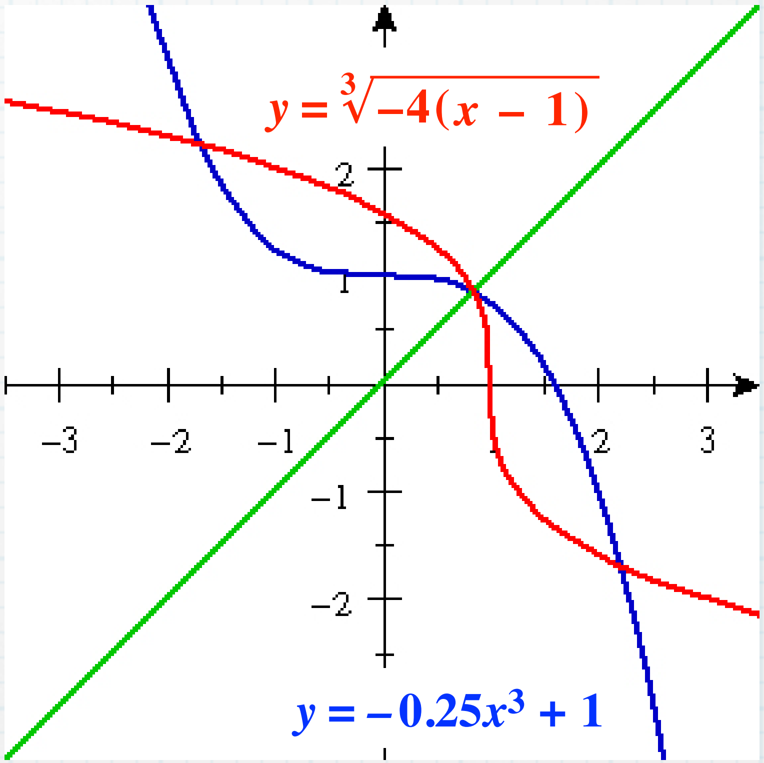
An Observation:



If a 1-to-1 cubic function is to pass through the points (a, b) and (b, a), its leading coefficient must be negative.

A Look at Some 1-to-1 Cubic Functions





Twin Theorem

Suppose f is an invertible function defined on $\mathbb{R}$. If f and f^{-1} intersect at (a, b), then they also intersect at (b, a).

Proof: If (a, b) is an intersection point, then $f(a) = b$ and $f^{-1}(a) = b$.

Since f and f^{-1} are inverses of each other, $a = f^{-1}(b)$ and $a = f(b)$.

Therefore (b, a) is also an intersection point.

Note that intersection points are either on the line $y = x$ or they have a twin.

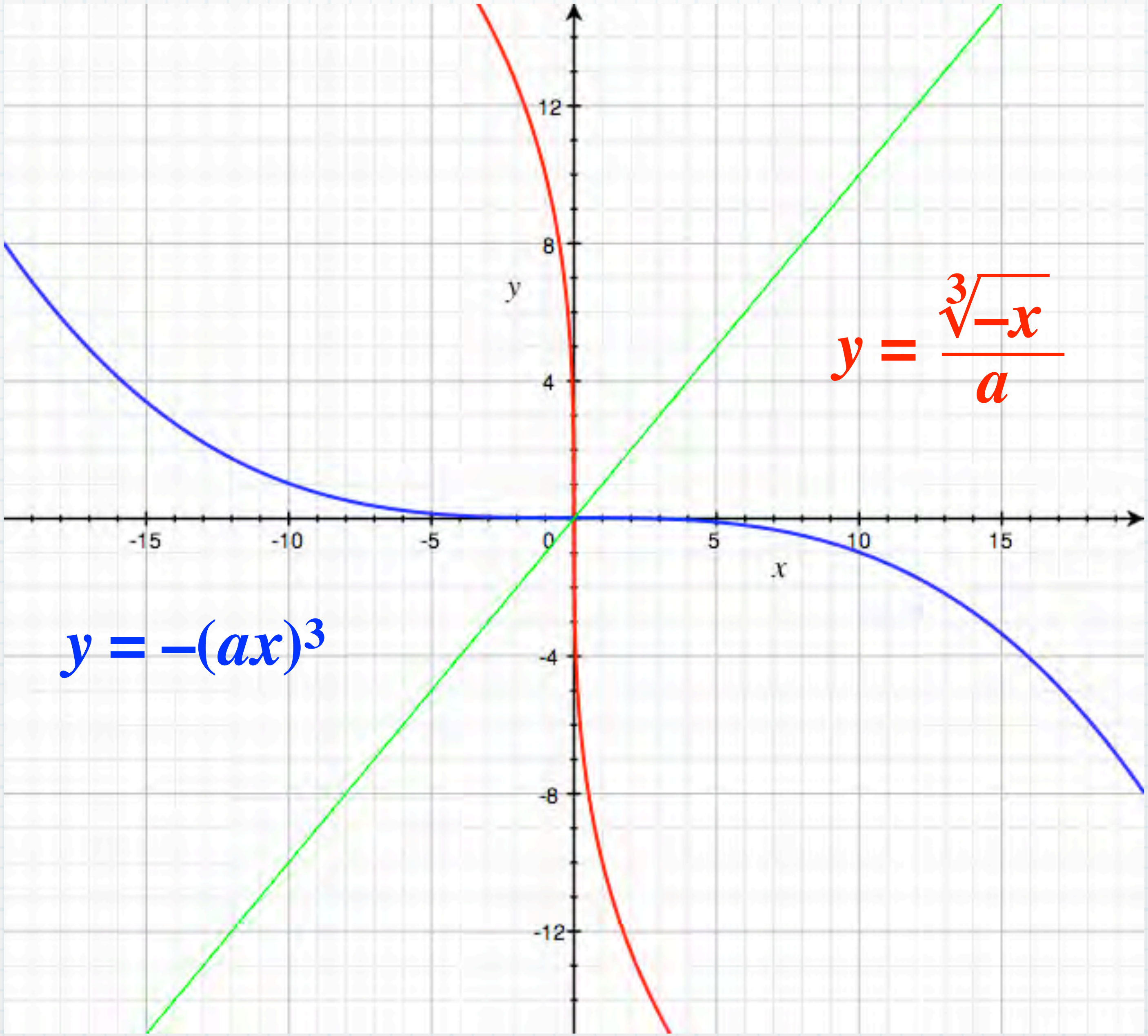
Recipes for More Functions that Intersect Off $y = x$

1. Throw in a parameter or two.

2. Use Composition of functions.

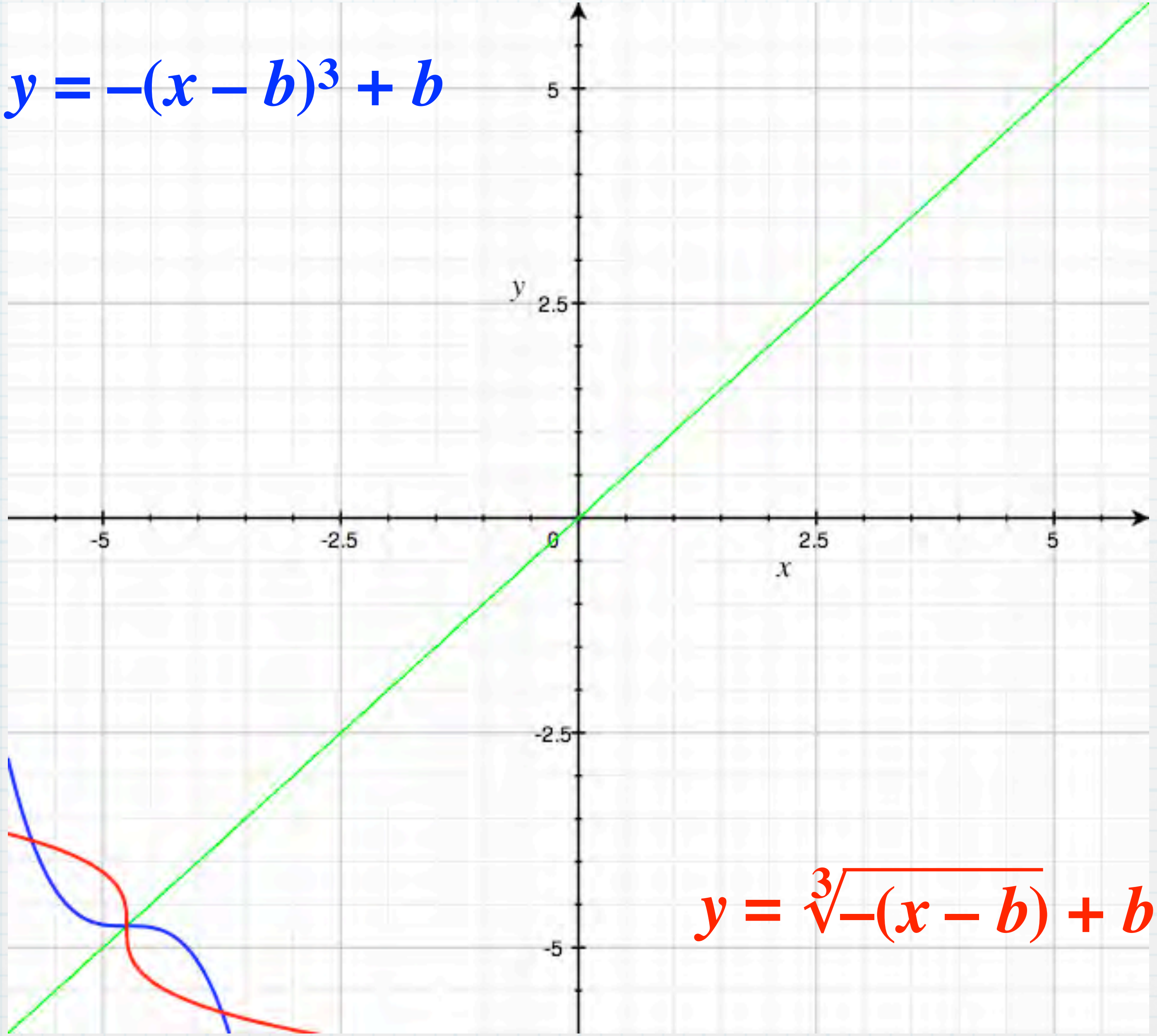
Letting a Parameter Vary

$a = 0.1 \dots 1.8$

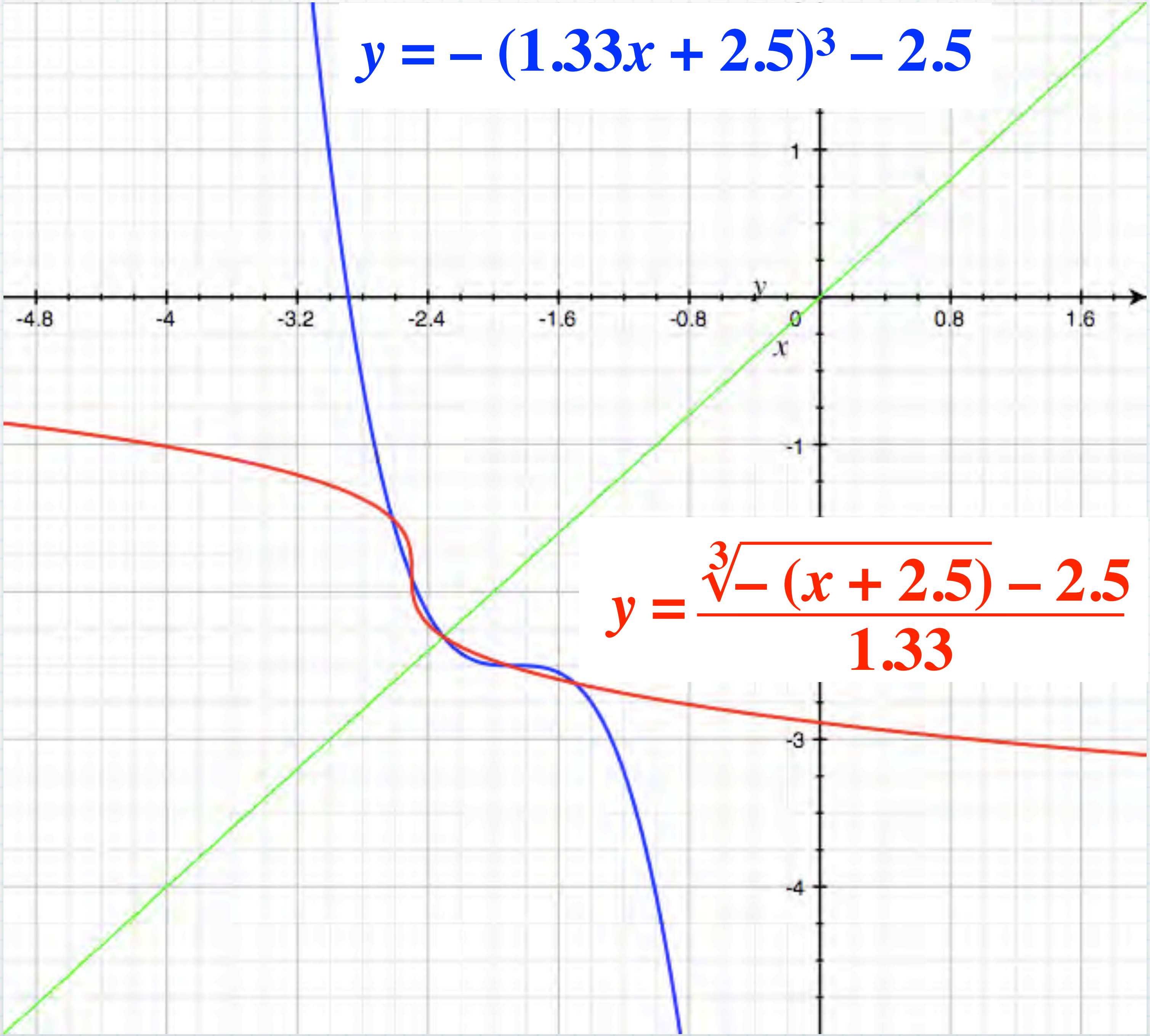


Letting a Parameter Vary

$b = -4.75 \ldots 4.75$



Isn't This Curious???



Using Composition

Suppose f is a decreasing function whose graph intersects f^{-1} at the point $(a, -a)$. Then $(-a, a)$ is also an intersection point (by the Twin Theorem.)

Let's evaluate the negation of the composition of f with itself at $x = a$ and at $x = -a$.

$$-[(f \circ f)(a)] = -f(f(a)) = -f(-a) = -a$$

$$-[(f \circ f)(-a)] = -f(f(-a)) = -f(a) = a$$

This negated composition function also passes through $(a, -a)$ and $(-a, a)$.

Furthermore

The function

$$-(f\circ f)(x)$$

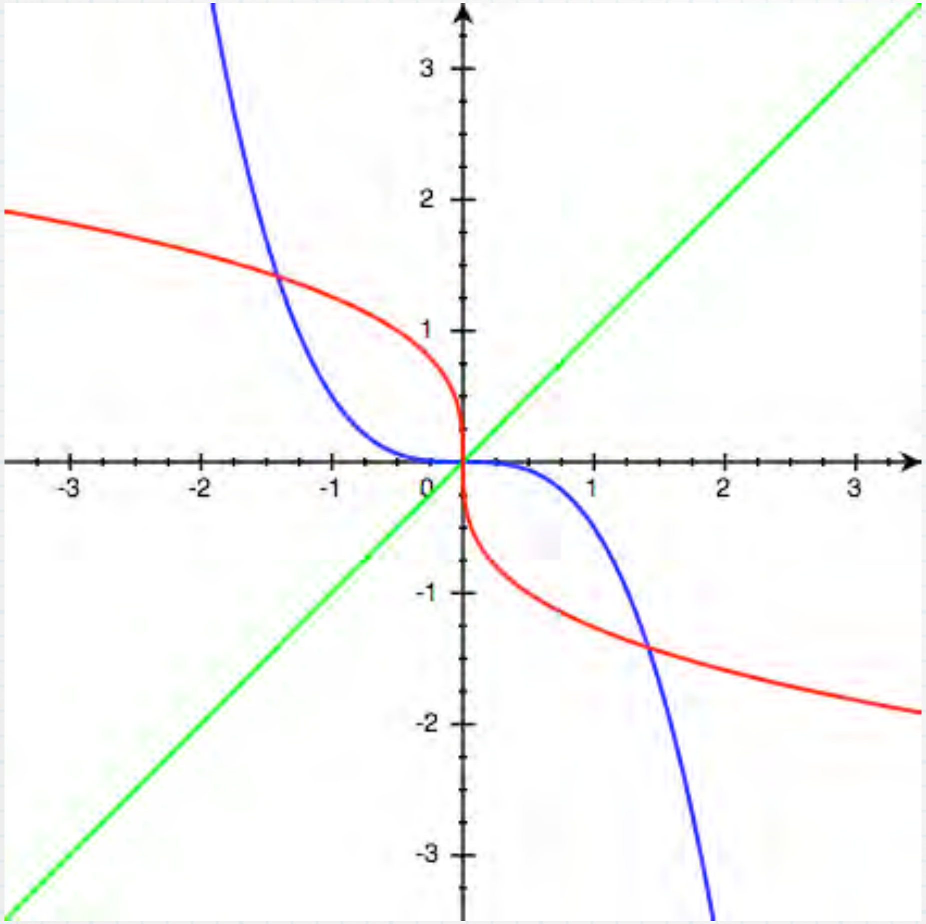
is also decreasing since

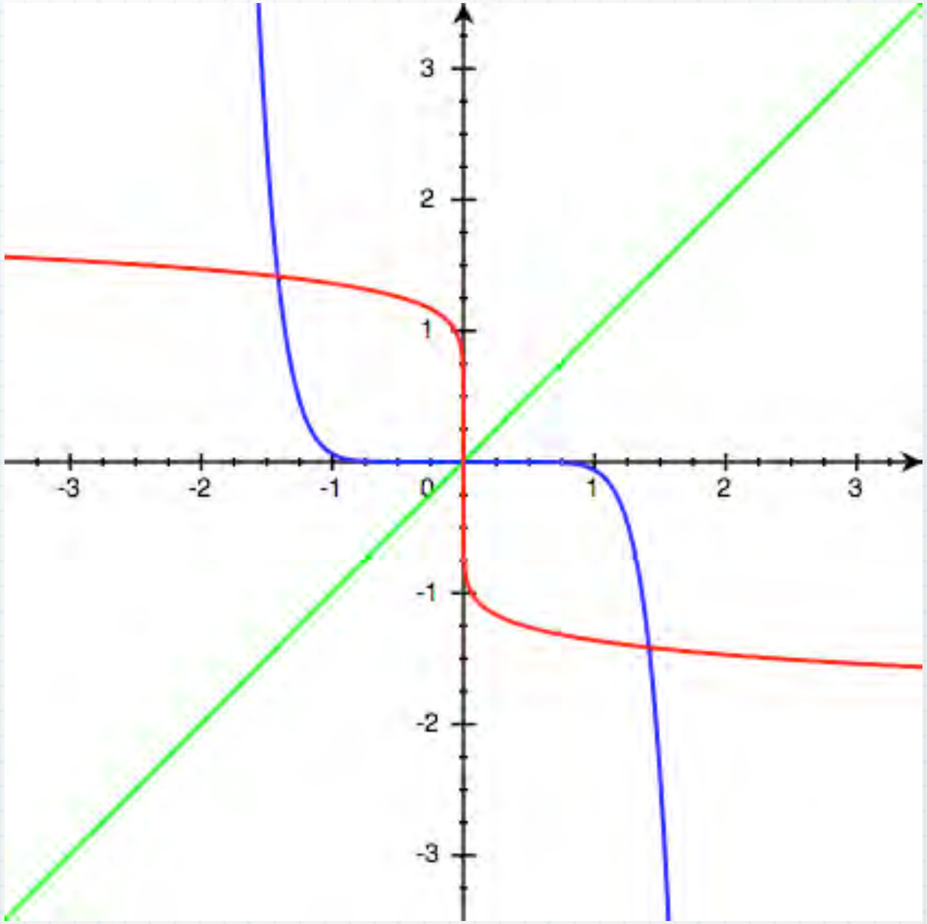
$$[-f(f(x))]' = -f'(f(x))\,f'(x).$$

$$(-)\;(-)\;(-)$$

That is, $-(f\circ f)(x)$ is one-to-one and is invertible.
It passes through $(a, -a)$ and $(-a, a)$, (previous slide)
and so does its inverse.

Composition Graphs





$y = -0.5x^3$

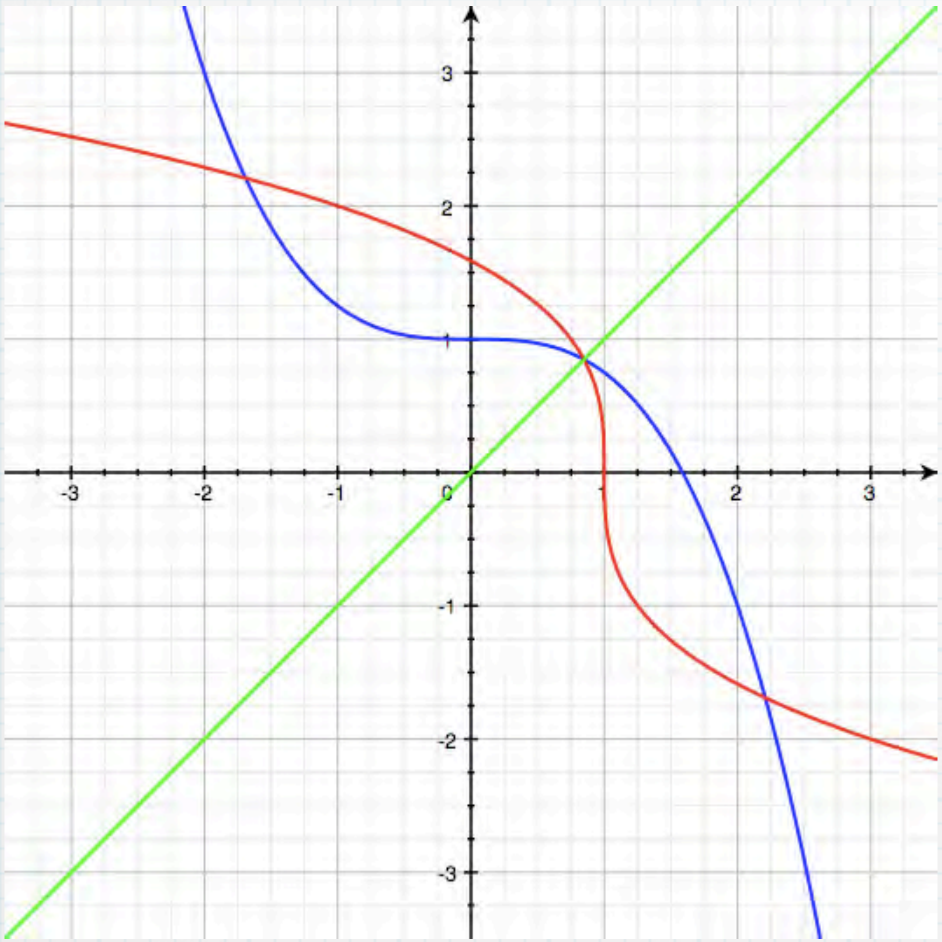
$x = -0.5y^3$

$y = -(-0.5(-0.5x^3)^3)$

$x = -(-0.5(-0.5y^3)^3)$

More Composition Graphs

(but I can't prove this)



$y = -0.25x^3 + 1$

$x = -0.25y^3 + 1$

$y = -(-0.25(0.25x^3 + 1)^3 + 1)$

$x = -(-0.25(0.25y^3 + 1)^3 + 1)$

Initial Results

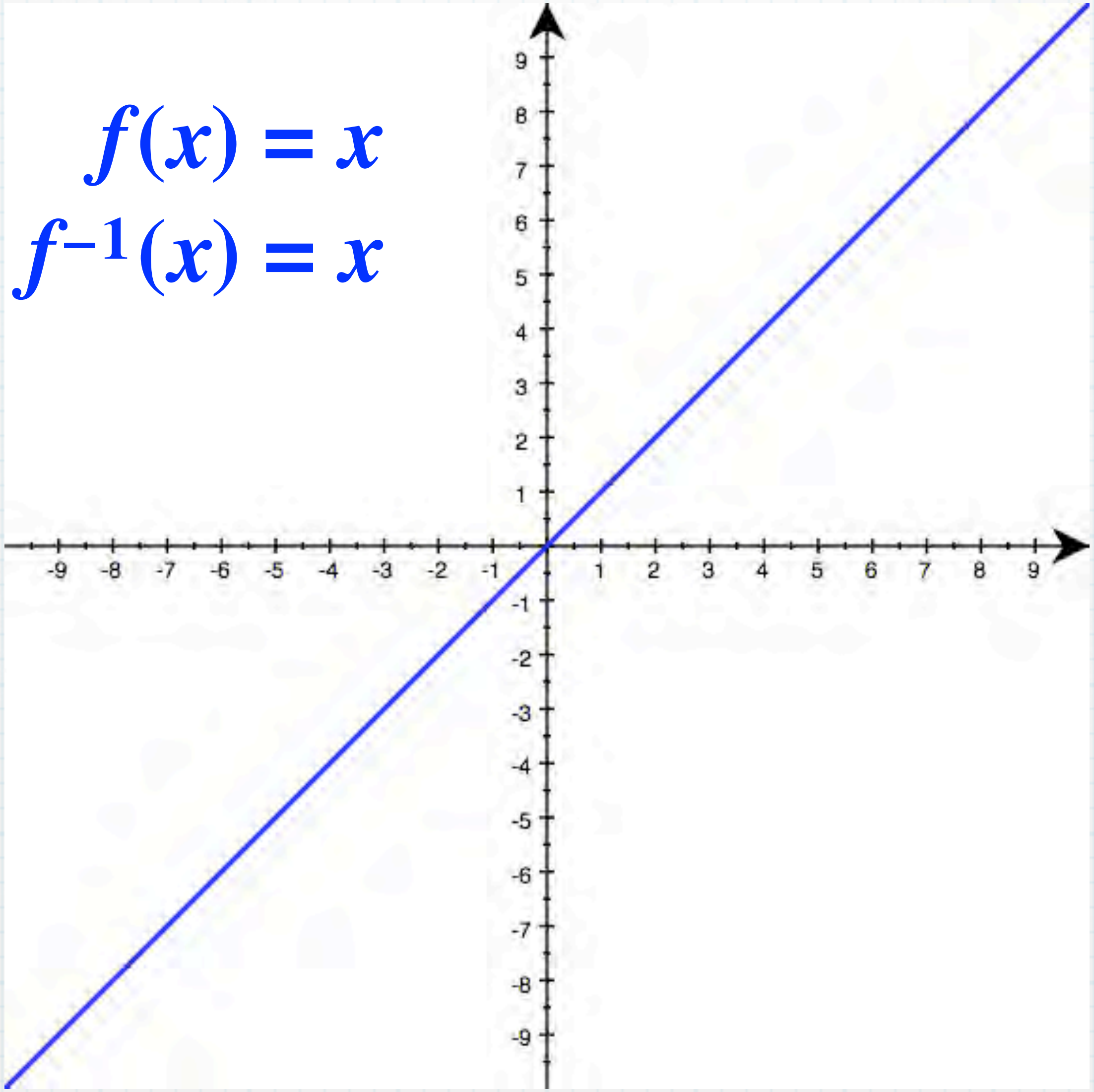
* Linear: Always intersect on $y = x$, except when the slope is -1 or 1.
* Quadratic: Intersection points that are not on $y = x$ are possible for a restricted domain.
* Cubic: Intersection points that are not on $y = x$ are possible for some 1-to-1 functions that have negative leading coefficients.
* Parameters and Composition of functions can be used to generate an endless supply.

Next

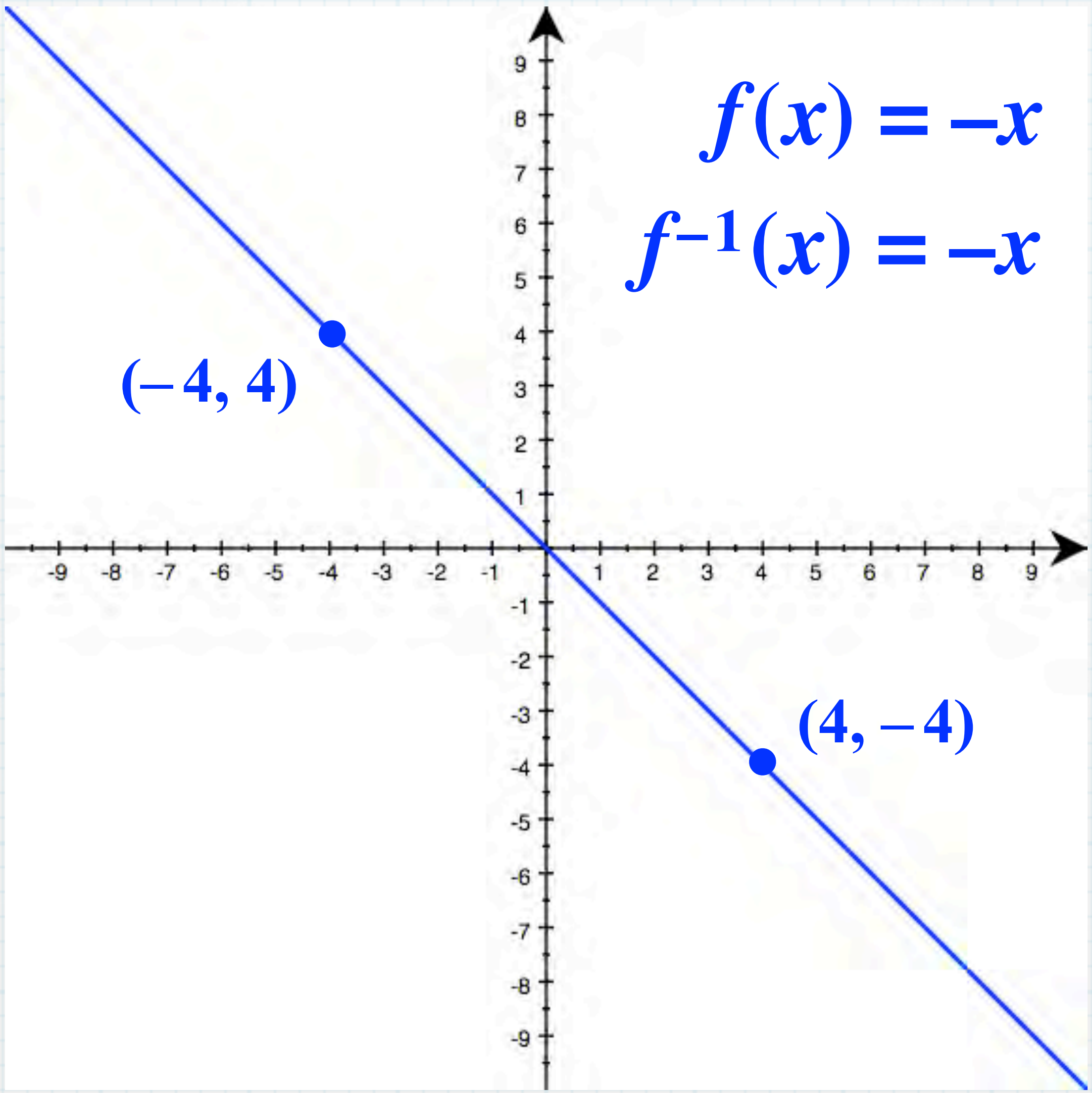
We will Investigate the Graphs of Functions and Their Inverses that:

* have an uncountably infinite number of intersection points.
* have a countably infinite number of intersection points.
* have a specified number of intersection points
* have no intersection points

Uncountably Infinite Intersections on the line $y = x$

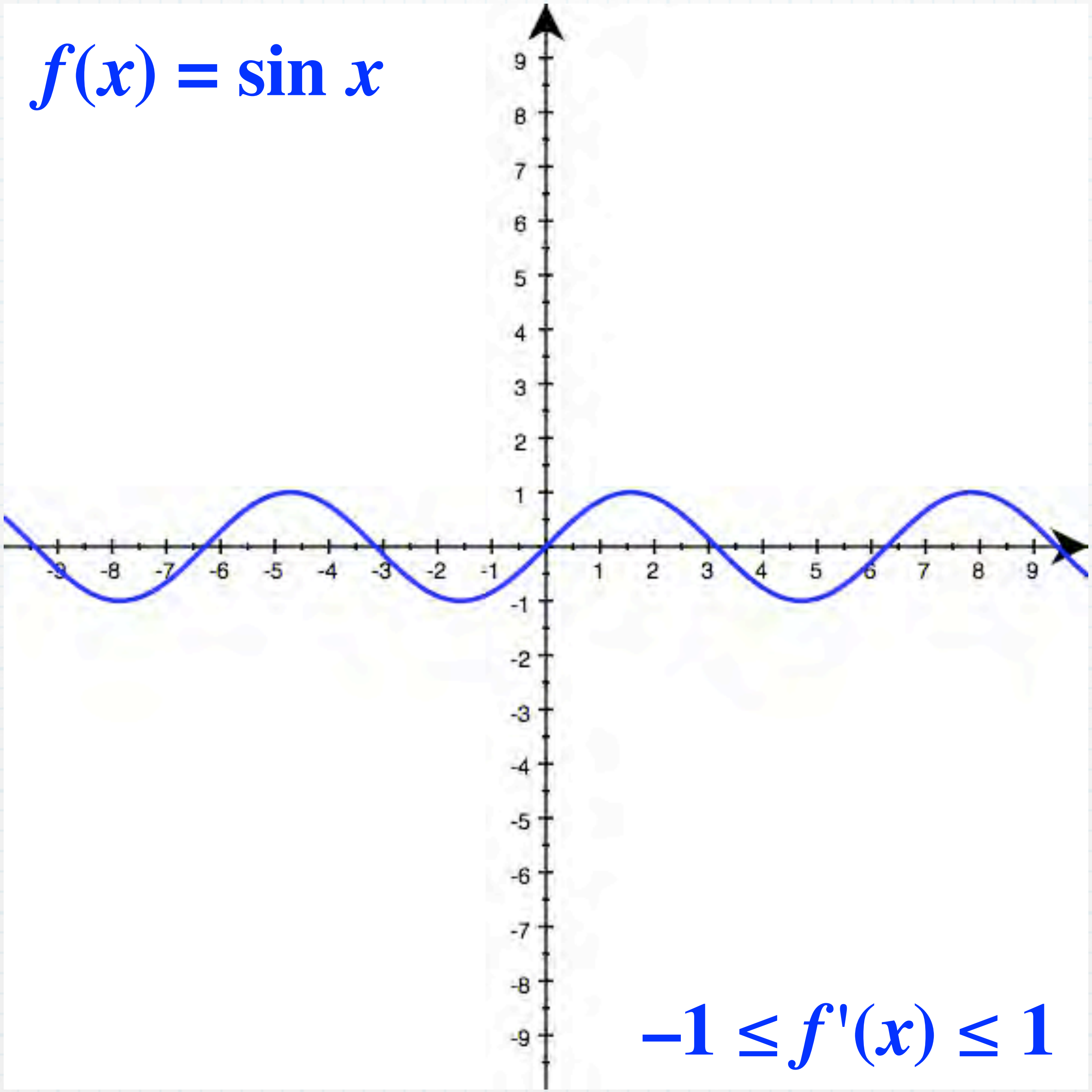


Uncountably Infinite Intersections NOT on the line $y = x$



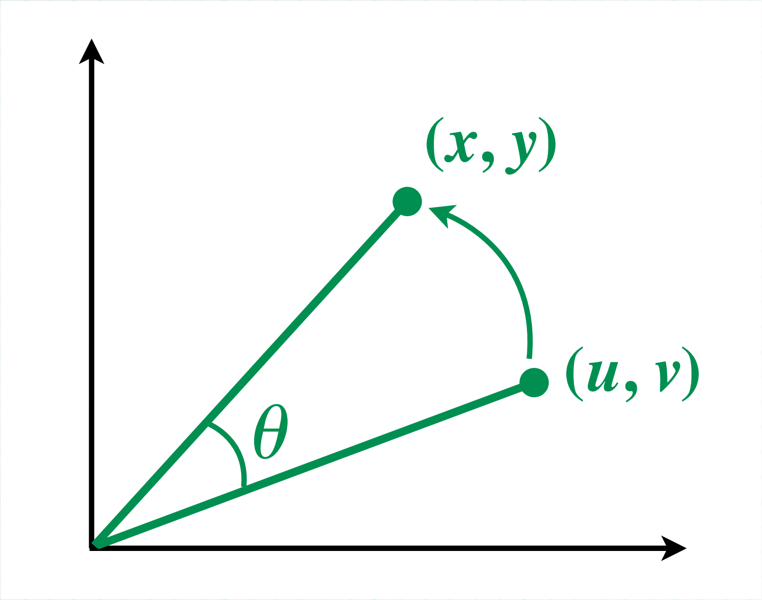
A Countably Infinite Number of Intersections

$f(x) = \sin x$



$-1 \leq f'(x) \leq 1$

Rotations



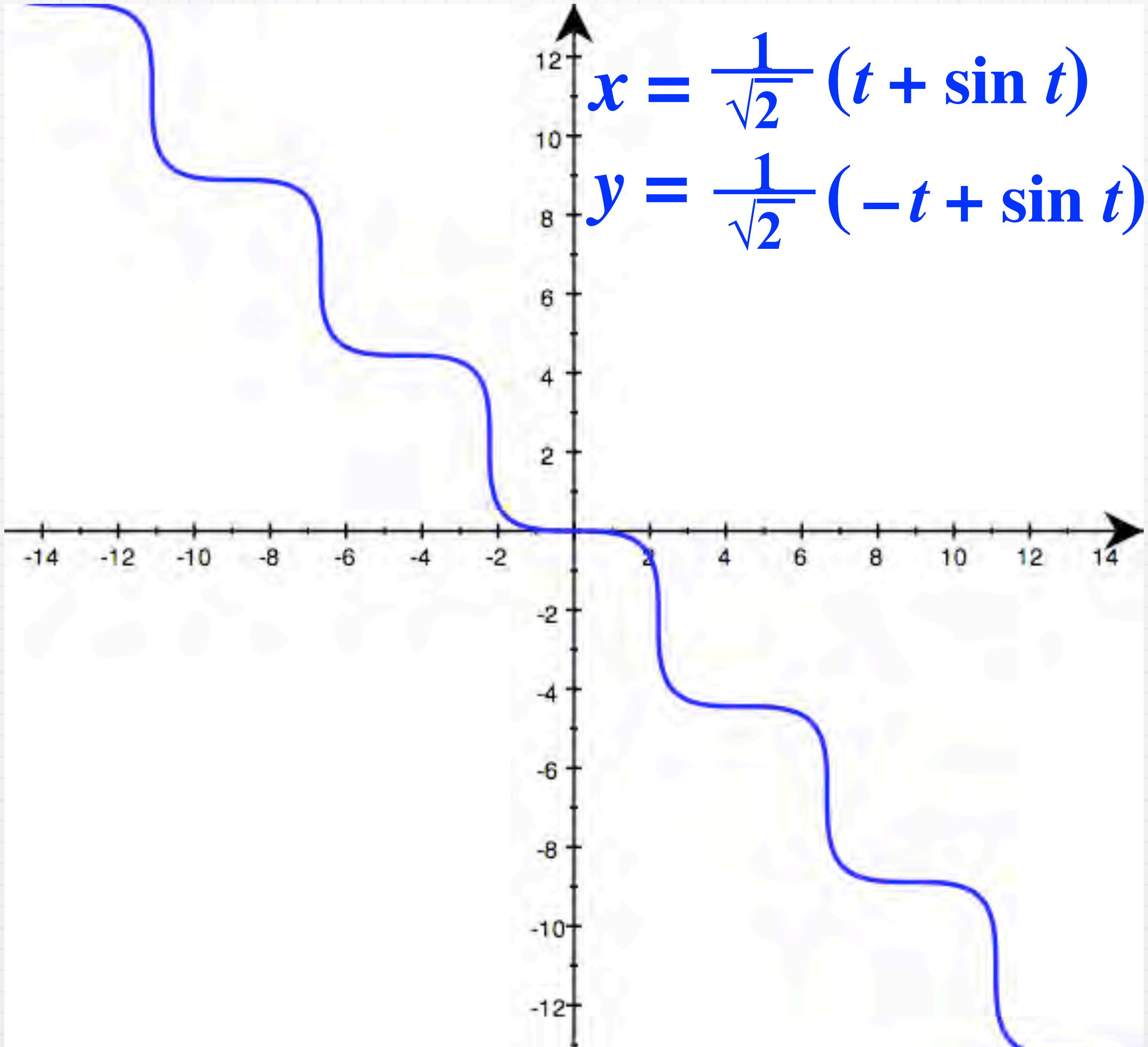
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

To rotate $f(x) = \sin x$ by $-45°$, we let

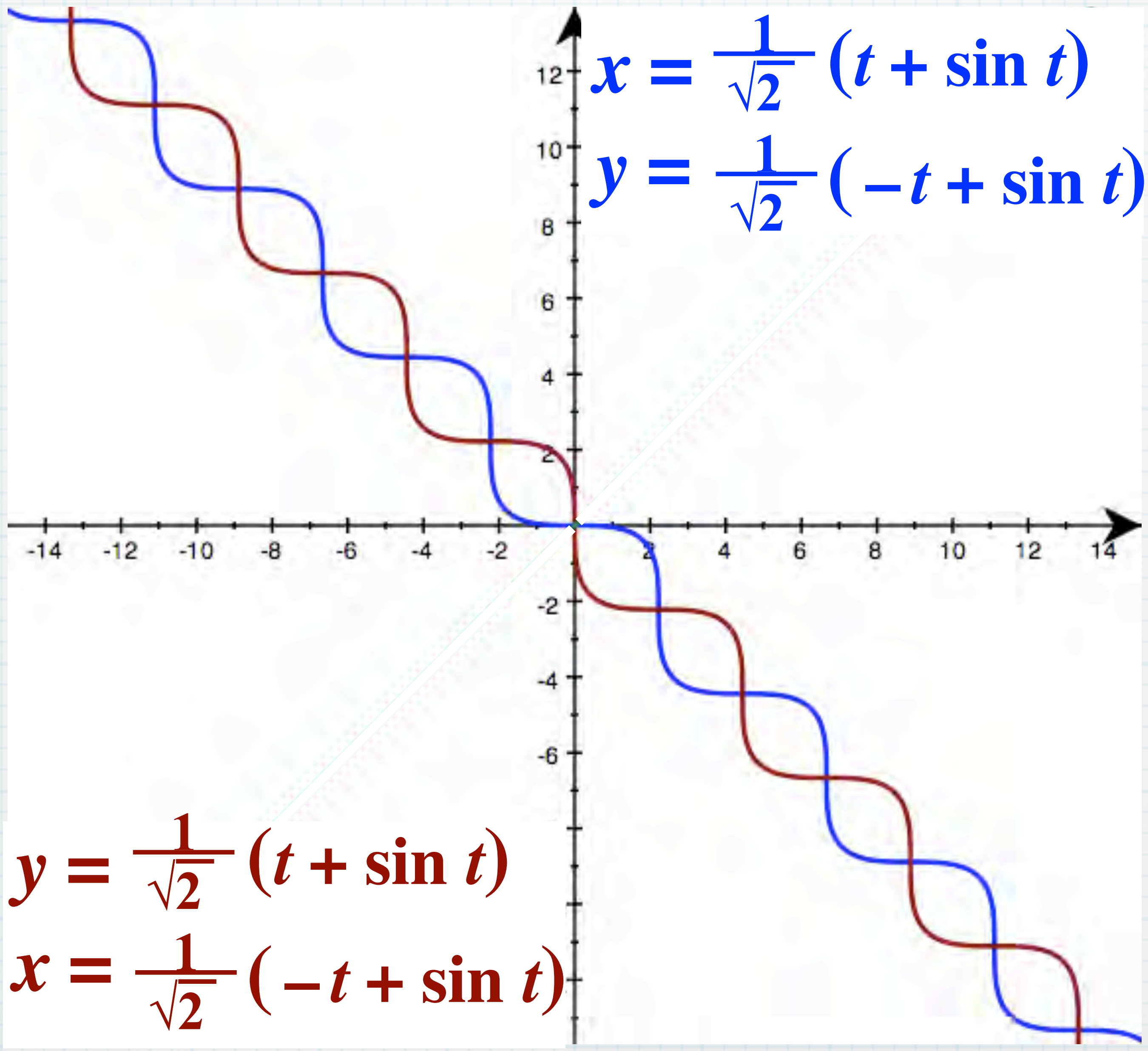
$\theta = -\dfrac{\pi}{4}$, $u = t$, and $v = \sin t$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} t \\ \sin t \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}(t + \sin t) \\ \frac{1}{\sqrt{2}}(-t + \sin t) \end{bmatrix}$$

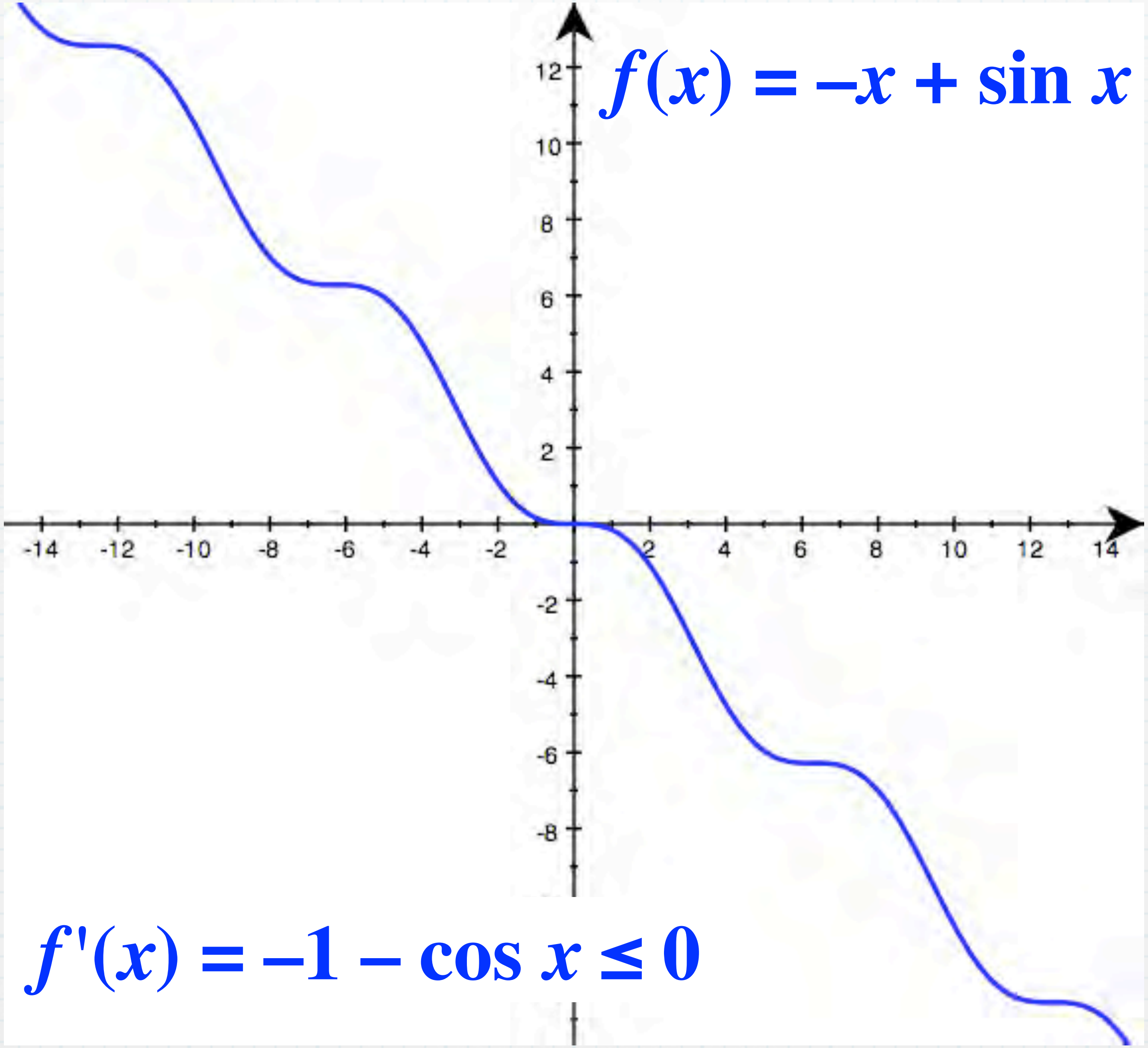
A Ballerina Step Function



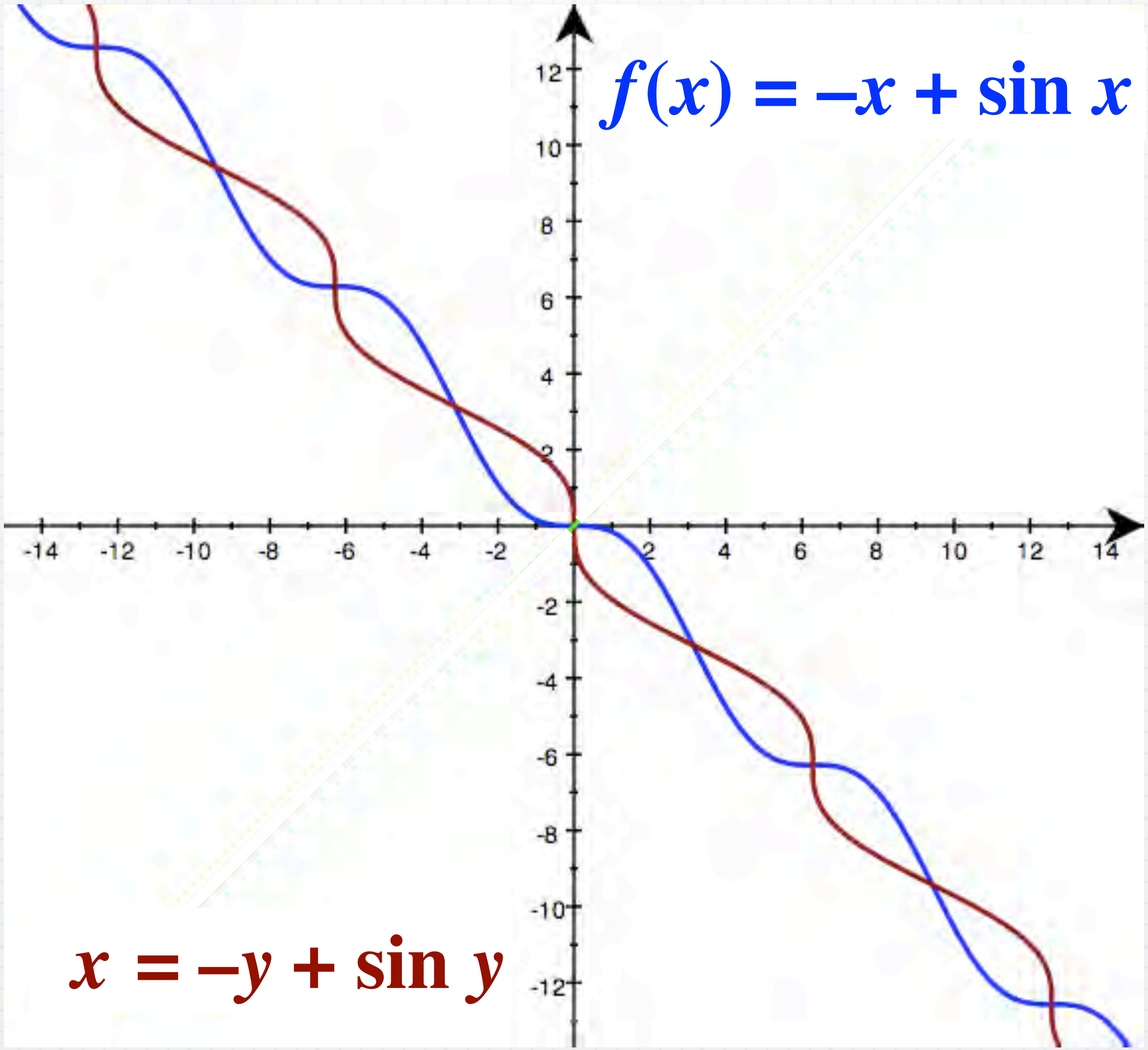
... and its Inverse



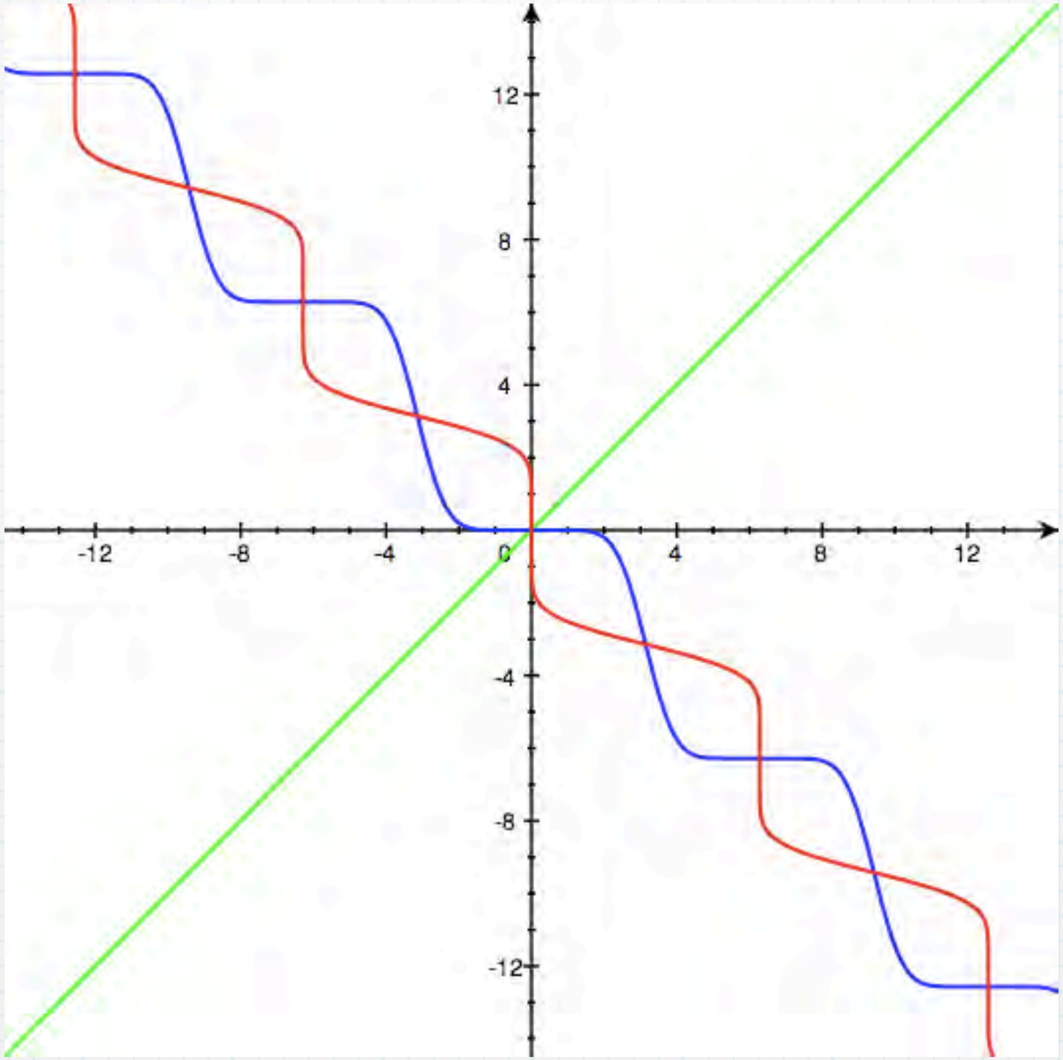
Using $\sin x$ as a Generating Function



… and its Inverse



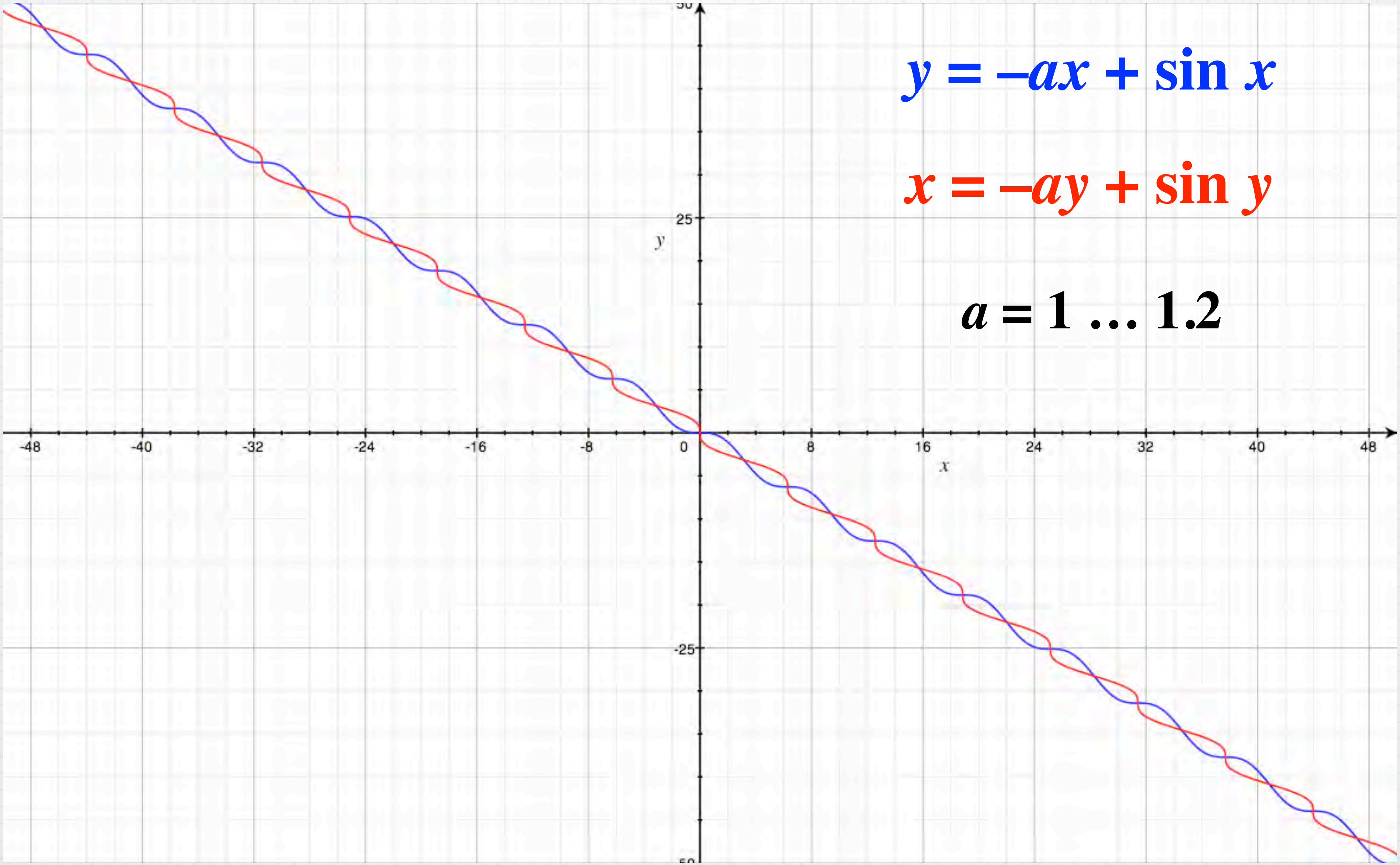
By the Way…



$$y = -(-(-x + \sin x) + \sin(-x + \sin x))$$

$$x = -(-(-y + \sin y) + \sin(-y + \sin y))$$

The Zipper Function



$$y = -ax + \sin x$$

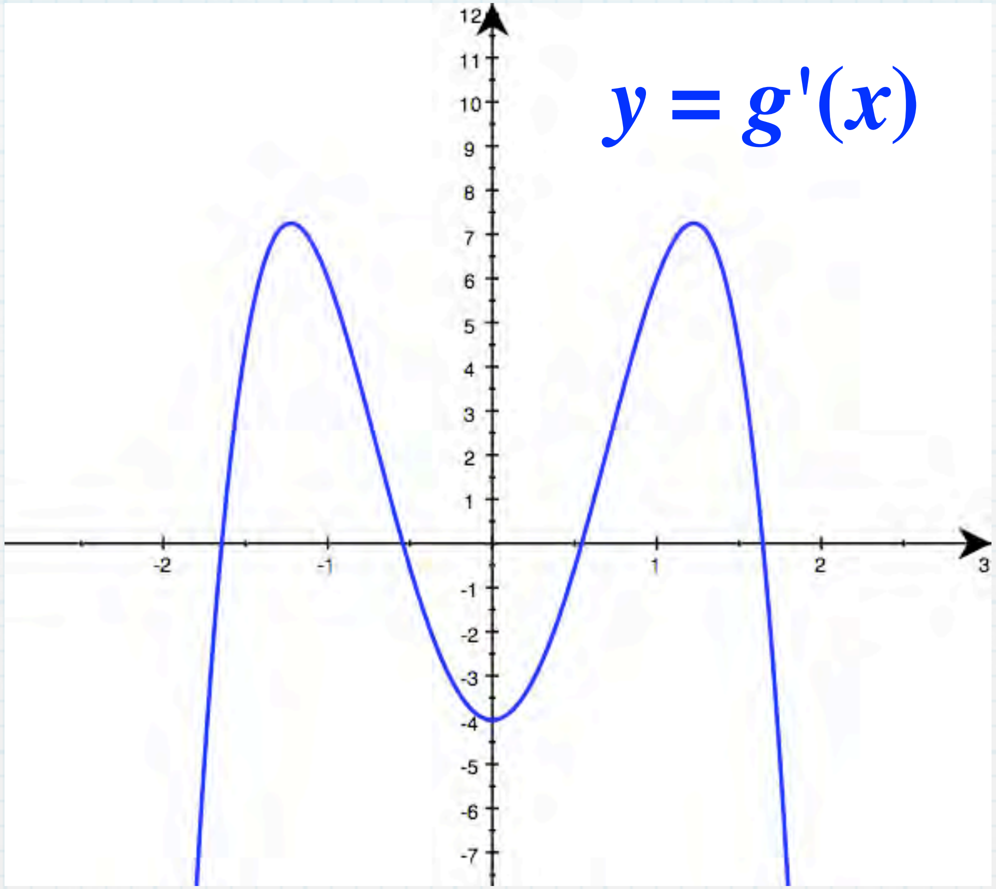
$$x = -ay + \sin y$$

$$a = 1 \ldots 1.2$$

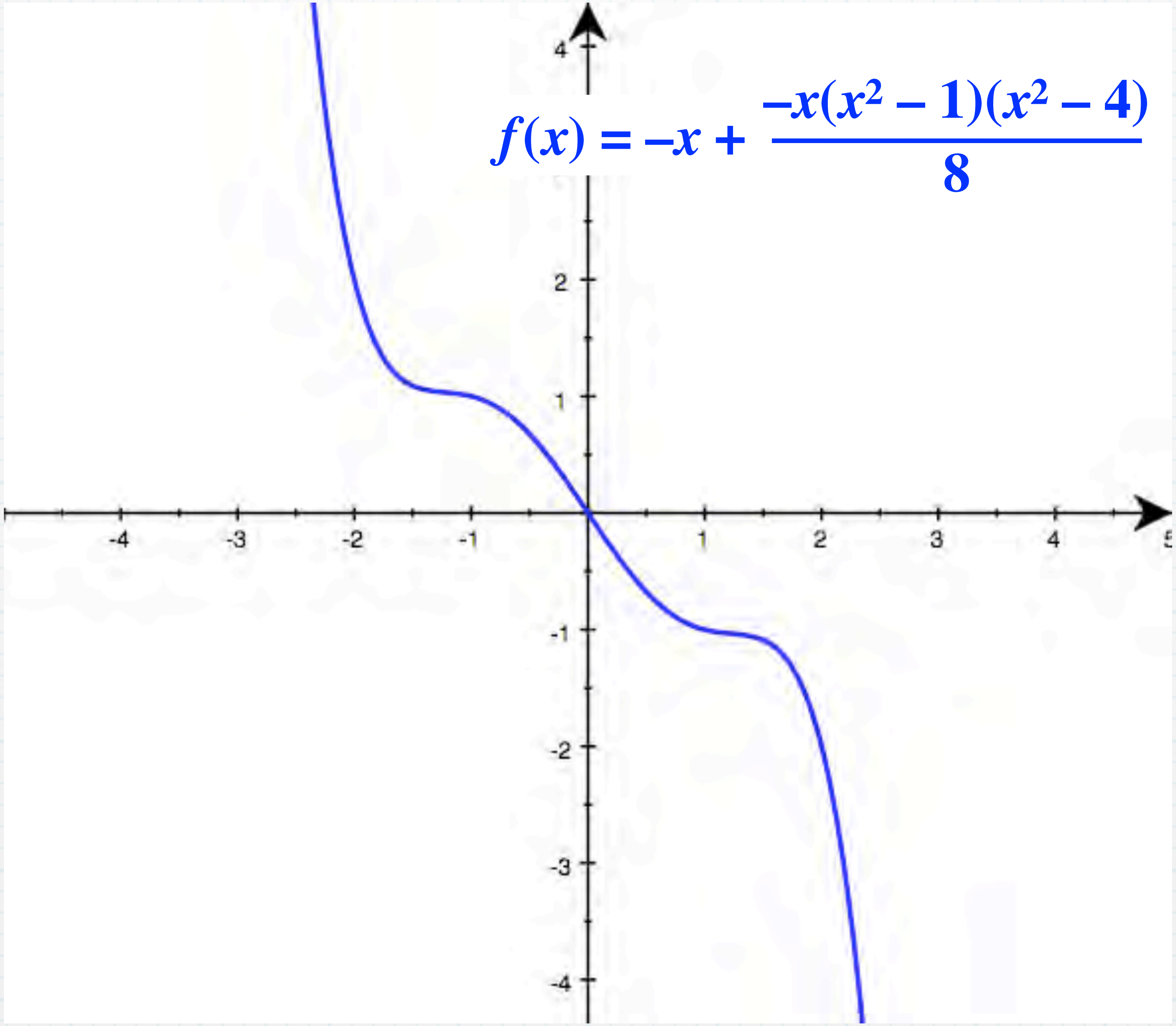
A Specified Odd Number of Intersection Points

Construction for 5 Intersection Points

$$g(x) = -x(x^2-1)(x^2-4) = -x^5 + 5x^3 - 4x$$
$$g'(x) = -5x^4 + 15x^2 - 4$$

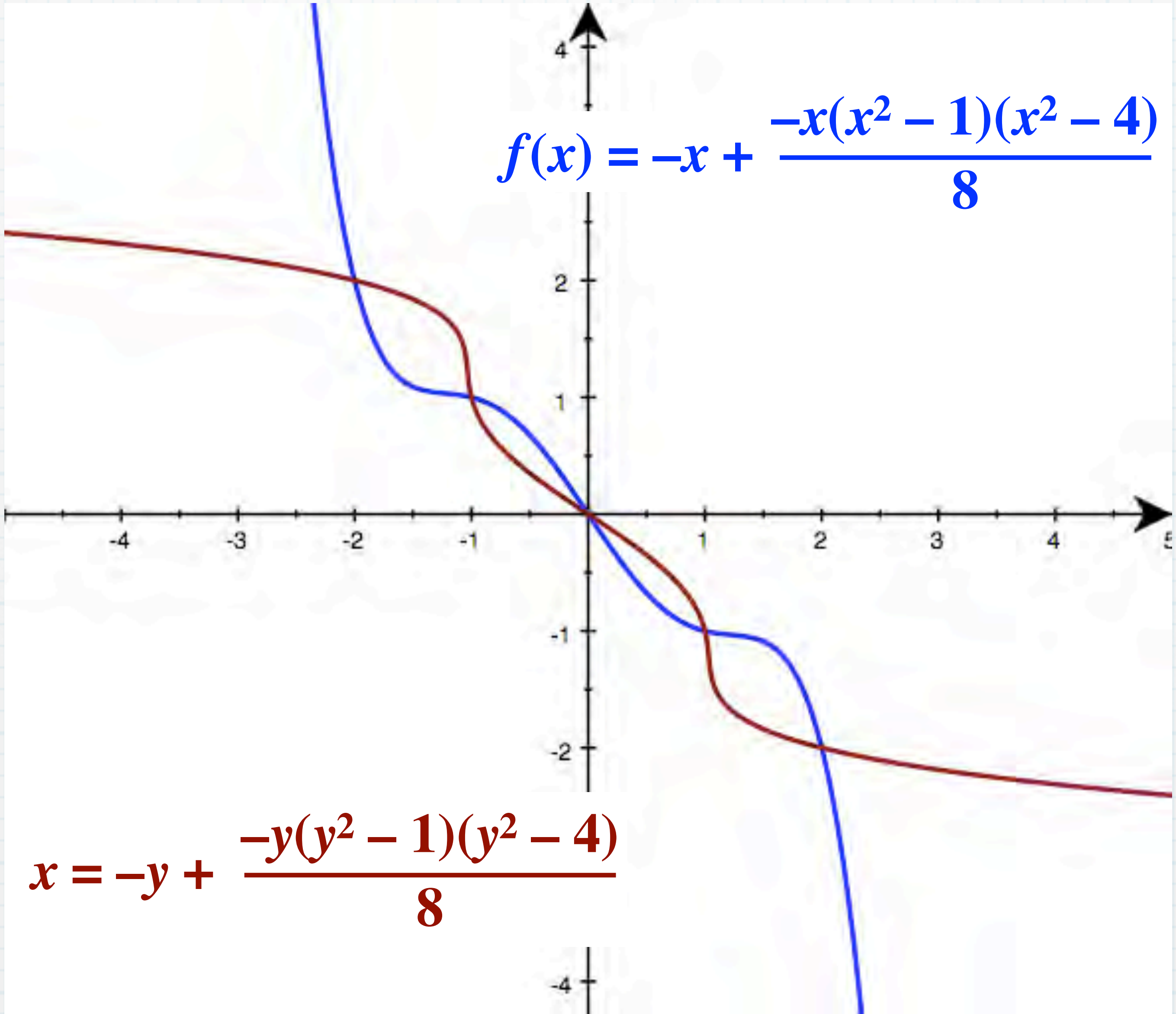


Note that $g'(x)$ is bounded above by 8.



Note that $f'(x) = -1 + \dfrac{g'(x)}{8} < 0.$

... and its Inverse



Corollary to the Twin Theorem

Suppose f is an invertible function defined on $\mathbb{R}$. If f and f^{-1} intersect at an even number of points, then f is increasing.

Proof by contradiction:

If f is decreasing, then f intersects the line $y = x$ at exactly one point. The remaining intersection points have twins.

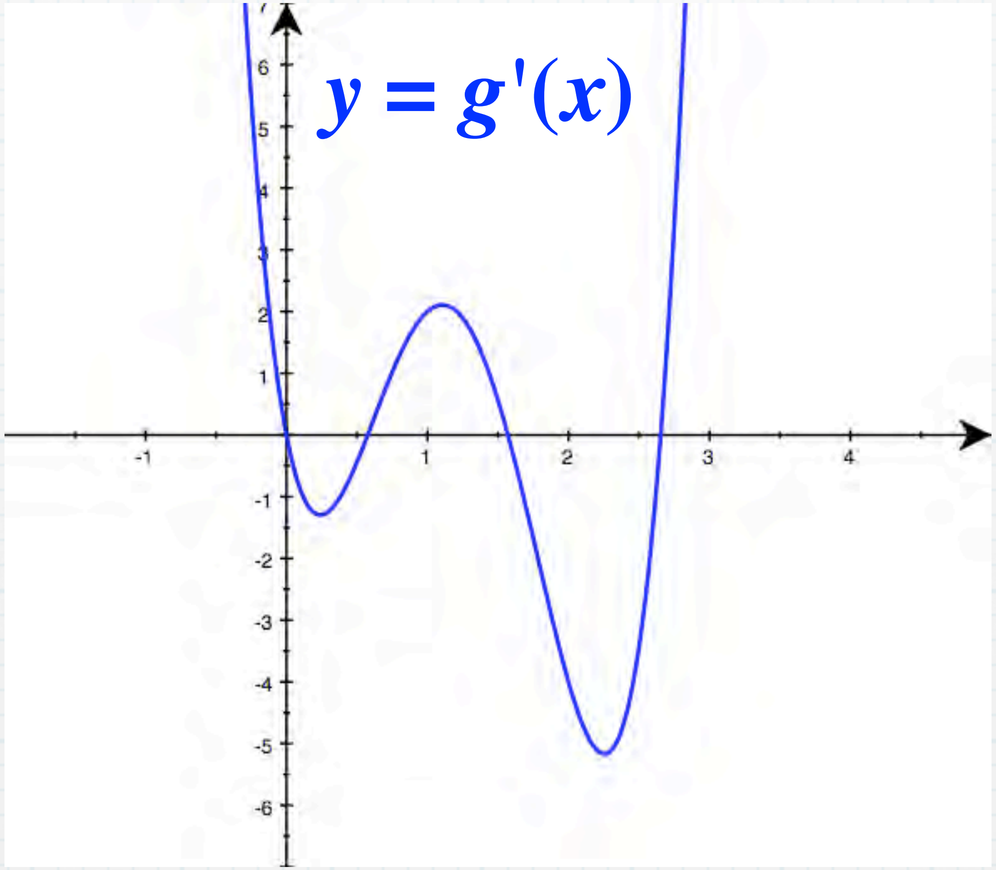
Hence f and f^{-1} intersect at an odd number of points.

A Specified Even Number of Intersection Points

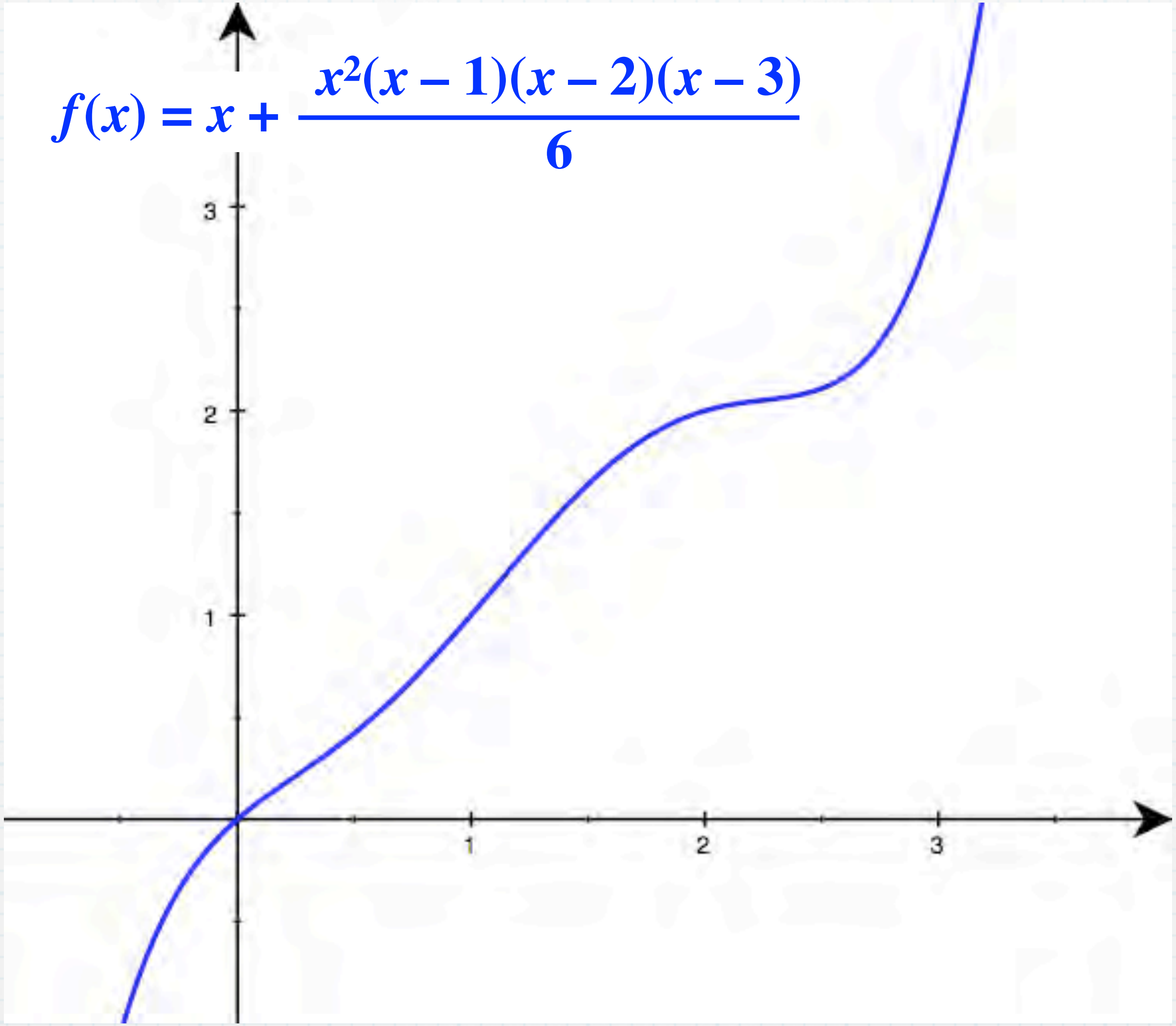
Construction for 4 Intersection Points

$$g(x) = x^2(x-1)(x-2)(x-3) = x^5 - 6x^4 + 11x^3 - 6x^2$$

$$g'(x) = 5x^4 - 24x^3 + 33x^2 - 12x$$

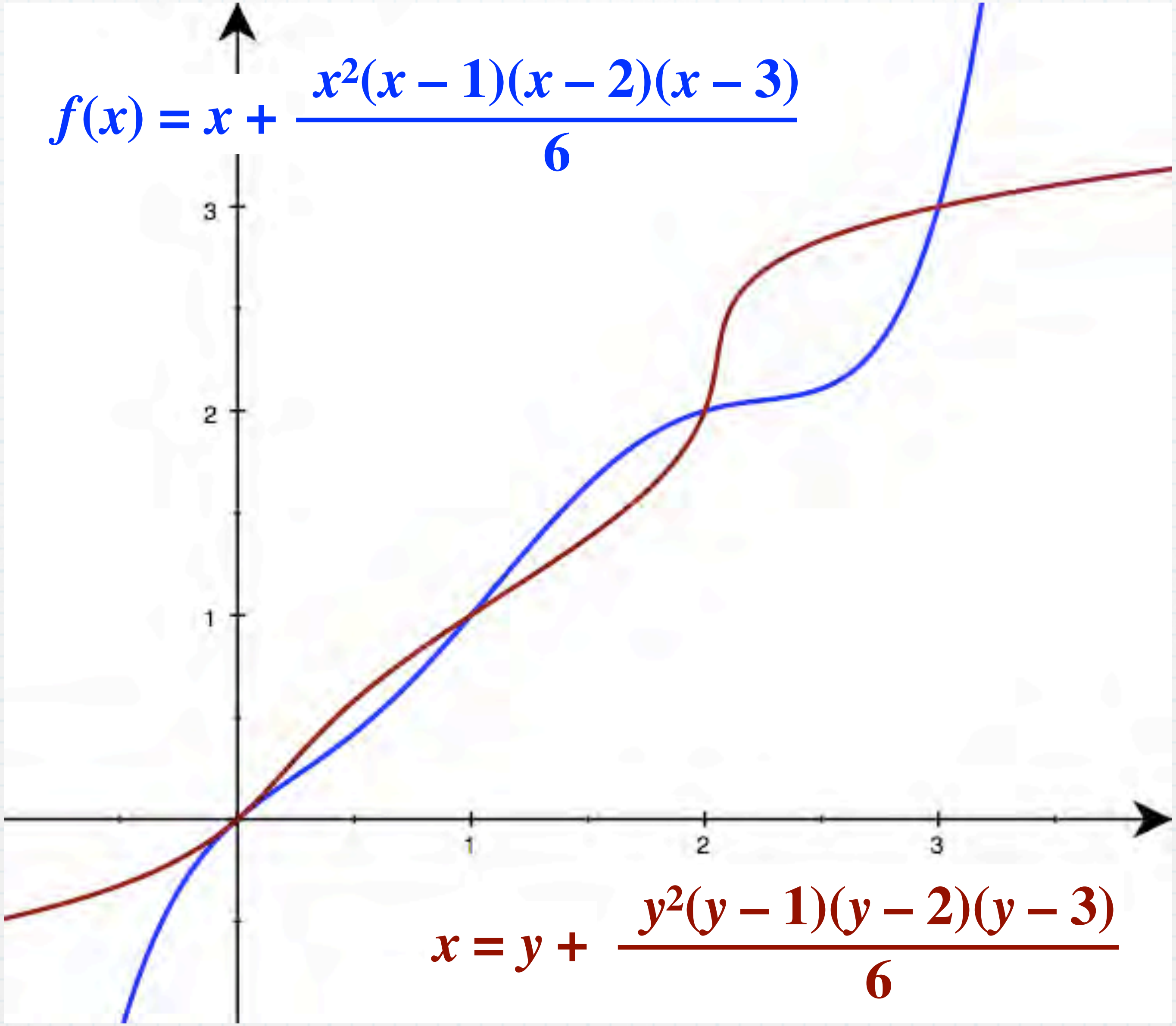


Note that $g'(x)$ is bounded below by −6.

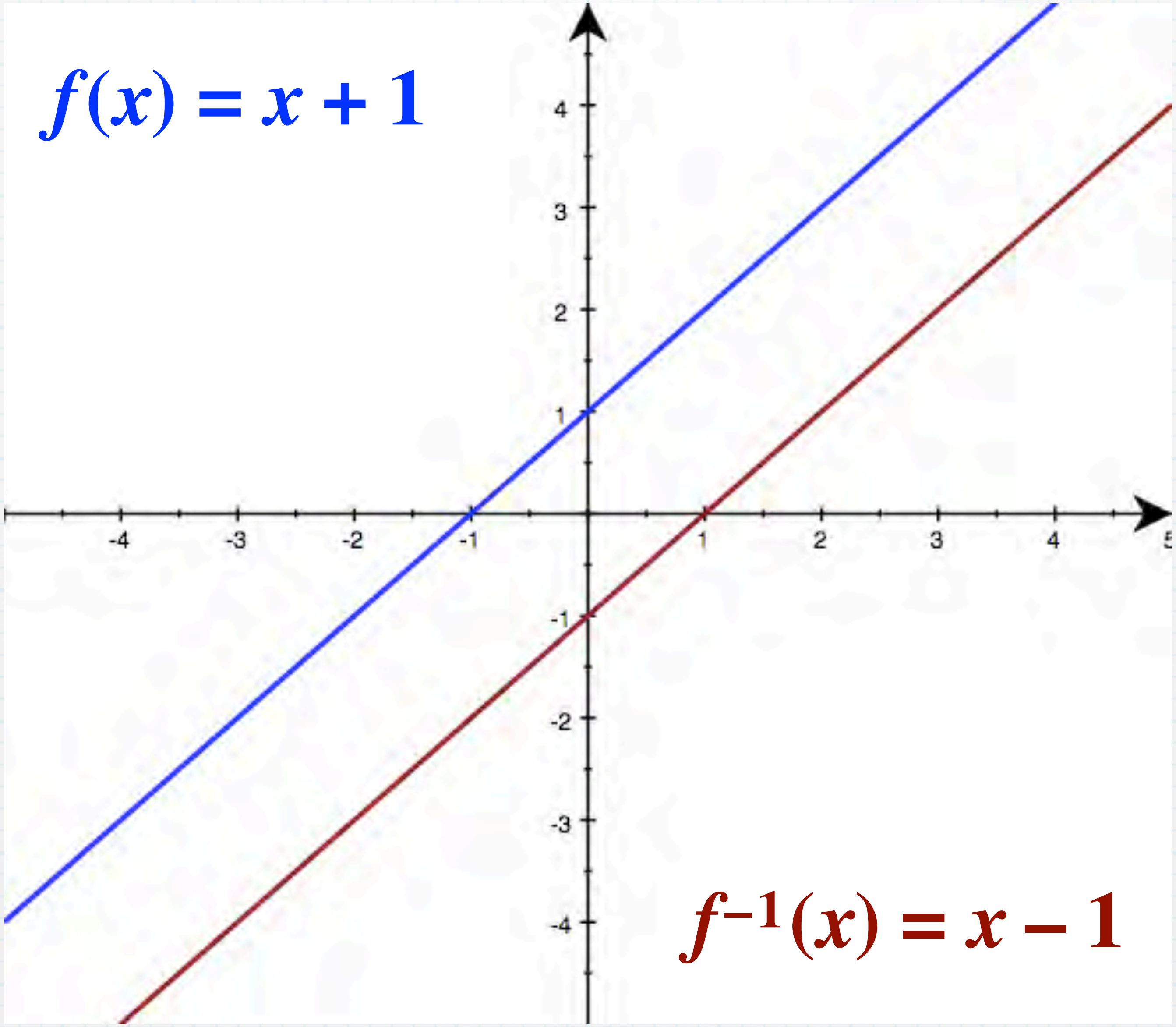


Note that $f'(x) = 1 + \dfrac{g'(x)}{6} > 0.$

… and its Inverse



No Intersection Points



More Interesting Functions

Exponential and Logarithmic Functions

Examples of Exponential Functions and Their Logarithmic Inverses

* $y = 10^x$ $\quad y = \log x$
* $y = 4^x$ $\quad y = \log_4 x$
* $y = e^x$ $\quad y = \ln x$
* $y = (1/3)^x$ $\quad y = \log_{1/3} x$

The Questions...

Let $b \neq 1$ represent a positive real number.

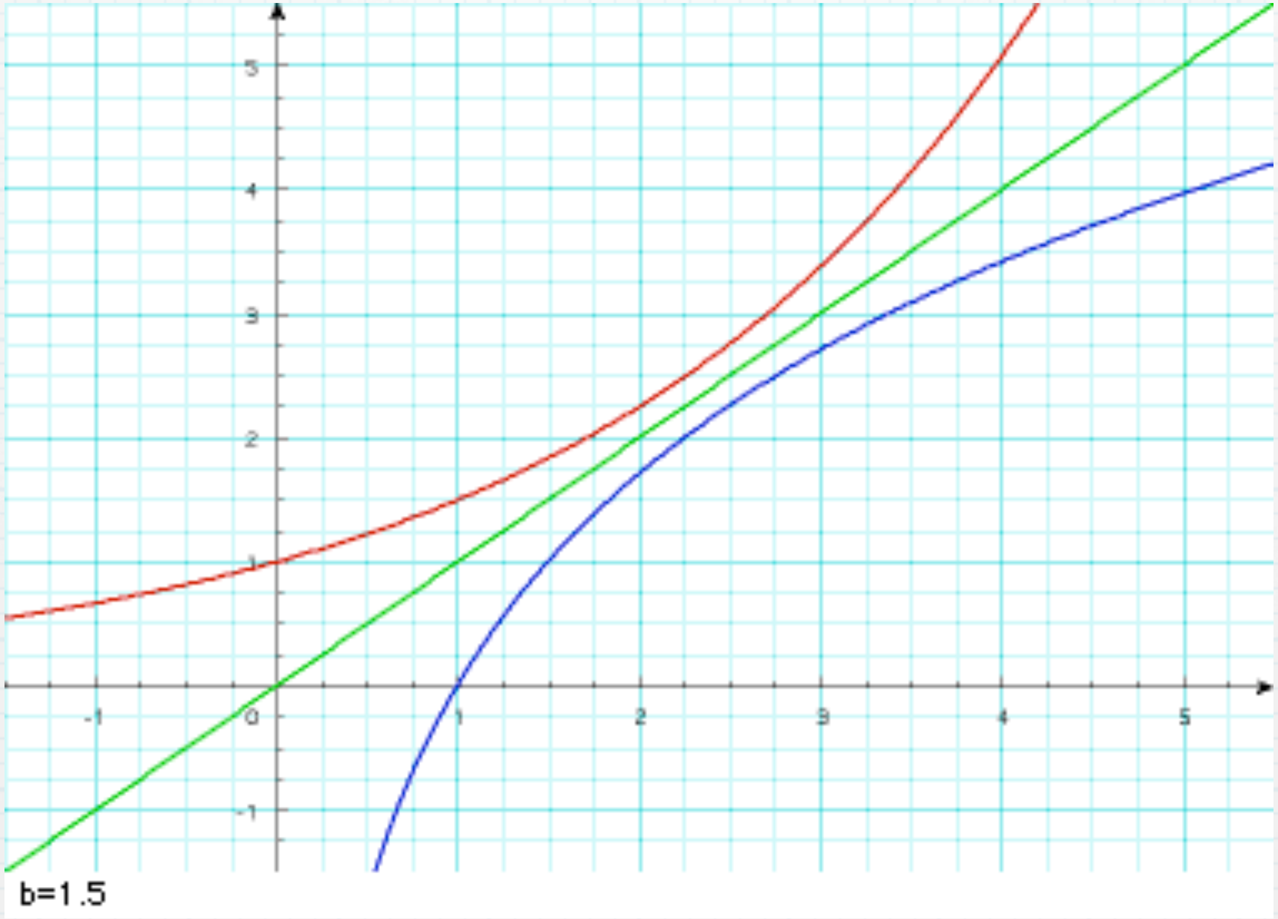
Will the graphs of

$y = b^x$ and $y = \log_b x$

ever intersect?

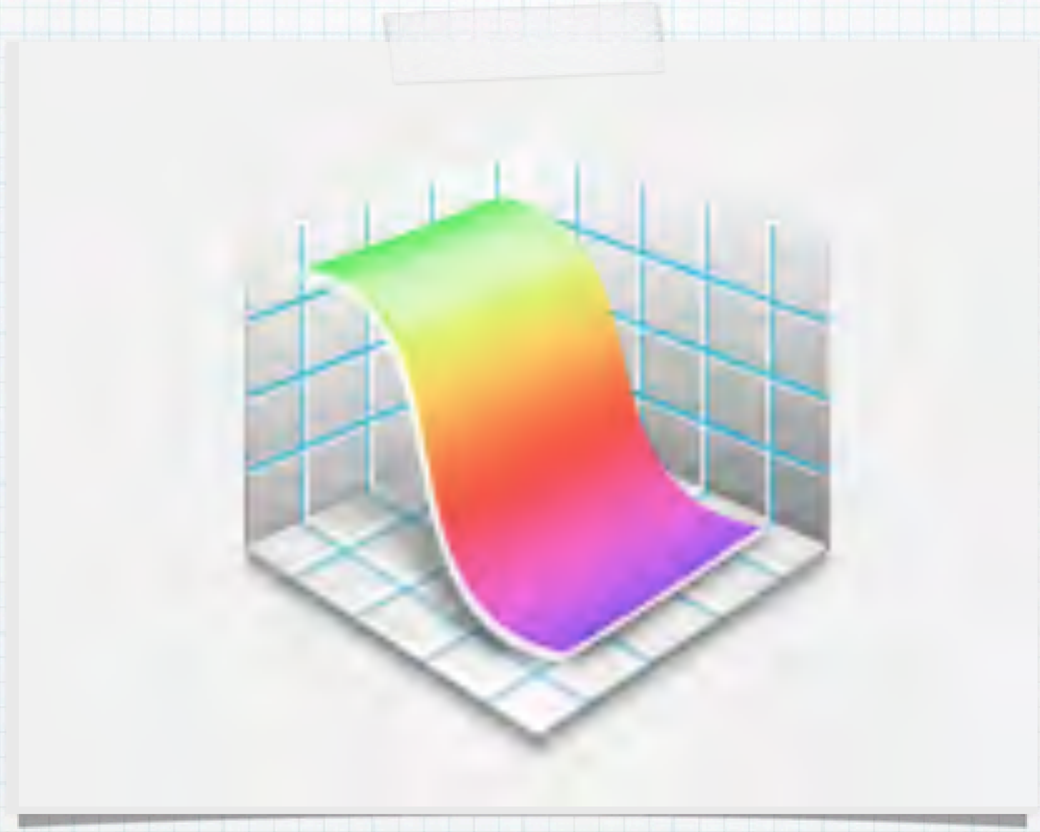
If so, what values of b make this possible?

When b varies from 1.5 to 8



If the graphs of an exponential function and its logarithmic inverse are to intersect, the following must be true:

The value of the base b is in the interval $(0, 1.5)$.



A Graphical Investigation

Is this intersection point (e, e)?

$x = e$ and $y = e$

$$y = b^x$$
$$e = b^e$$
$$\ln e = \ln b^e$$
$$1 = e \ln b$$
$$1/e = \ln b$$
$$e^{1/e} = e^{\ln b}$$
$$b = e^{1/e}$$

$$y = \log_b x$$
$$e = \log_b e$$
$$b^e = b^{\log_b e}$$
$$b^e = e$$
$$\vdots$$
$$b = e^{1/e}$$

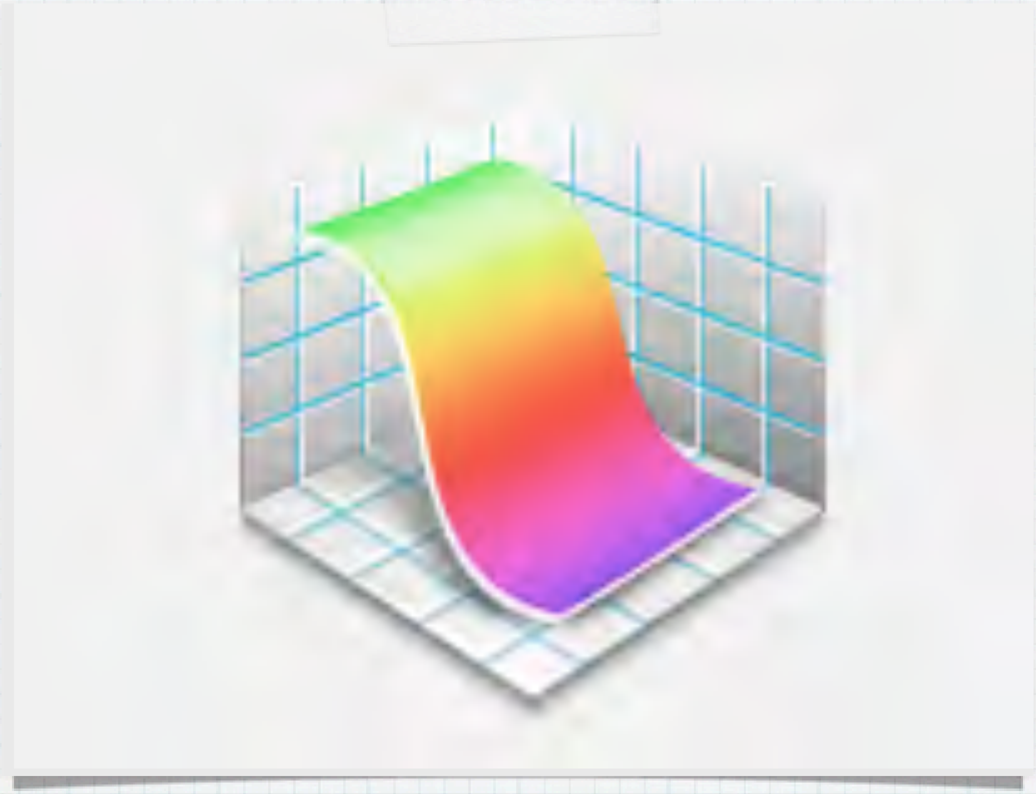
Note: $e^{1/e} \approx 1.444667861$

Our First Observation:

The graphs of $y = b^x$ and $y = \log_b x$ have two intersection points on the line $y = x$ when b is in the interval $(1, e^{1/e})$. Furthermore, they intersect at the point (e, e) when $b = e^{1/e}$.

The Next Question...

What happens when $0 < b < 1$?



A Graphical Investigation

What are these Numbers?

- When $b \approx 0.066$ there is an intersection point at approximately $(0.368, 0.368)$.
- Interesting coincidence #1: $e^{-e} \approx 0.065988$
- Interesting coincidence #2: $e^{-1} \approx 0.367879$

Is this intersection point (e^{-1}, e^{-1})?

$x = e^{-1}$ and $y = e^{-1}$

$$y = b^x$$
$$e^{-1} = b^{e^{-1}}$$
$$\ln e^{-1} = \ln b^{e^{-1}}$$
$$-1 = e^{-1} \ln b$$
$$-e = \ln b$$
$$e^{-e} = e^{\ln b}$$
$$b = e^{-e}$$

$$y = \log_b x$$
$$e^{-1} = \log_b e^{-1}$$
$$b^{e^{-1}} = b^{\log_b e^{-1}}$$
$$b^{e^{-1}} = e^{-1}$$
$$\vdots$$
$$b = e^{-e}$$

The Results

* The graphs intersect 3 times for b in $(0, e^{-e})$.
* The graphs intersect 1 time for b in $[e^{-e}, 1)$.
* The graphs intersect 2 times for b in $(1, e^{1/e})$.
* The graphs intersect 1 time when $b = e^{1/e}$.
* The graphs don't intersect for b in $(e^{1/e}, \infty)$.

Futhermore,
when $b = e^{-e}$ the intersection point is (e^{-1}, e^{-1})
and when $b = e^{1/e}$ the intersection point is (e, e).

For Future Discussion

* If we plot those intersection points that are not on $y = x$, what type of regression would give the best fit?
* What about rational functions?
* When can the formula for the inverse be written explicitly?

Thanks for coming!